Logistic Growth

$$\frac{dy}{dt} = ky(L - y)$$

where L is the carrying capacity (upper bound) and k is the constant of proportionality.
If you solve the differential equation using very tricky separation of variables, you get the general solution:

$$y = \frac{L}{1 + ce^{-Lkt}}$$

EX#1 :

A highly contagious "pinkeye" (scientific name: Conjunctivitus itchlikecrazius) is ravaging the local elementary school. The population of the school is 900 (including students and staff), and the rate of infection is proportional both to the number infected and the number of students whose eyes are pus-free. If seventy-five people were infected on December 15 and 250 have contracted pinkeye by December 20, how many people will have gotten the gift of crusty eyes by Christmas Day?

Solution

Because of the proportionality statements in the problem, logistic growth is the approach we should take. The upper limit for the disease will be L = 900; it is impossible for more than 900 people to be infected since the school only contains 900 people. This gives us the equation

$$y = \frac{900}{1 + ce^{-900kt}}$$

We will interpret $t = 0$ as December 15, since that is the earliest information given. Therefore, we know that $y(0) = 75$. Plug that information into the equation.

$$75 = \frac{900}{1 + ce^{0}}$$

$$75 + 75c = 900$$

$$c = \frac{825}{75} = 11$$

Five days later, 250 people have contracted pinkeye, so plug that information (and the c we just found) to find k:

$$250 = \frac{900}{1 + 11e^{-900k \cdot 5}} \quad \Rightarrow \quad 250(1 + 11e^{-4500k}) = 900$$

$$11e^{-4500k} = \frac{13}{5} \quad \Rightarrow \quad e^{-4500k} = \frac{13}{55}$$

$$\ln e^{-4500k} = \ln\left(\frac{13}{55}\right) \quad \Rightarrow \quad -4500k = -1.442383838$$

$$k = 0.0003205297$$

Finally, we have the equation $y = \frac{900}{1 + 11e^{-900(0.0003205297)t}}$. We want to find the number of infections on December 25, so $t = 10$.

$$y = \frac{900}{1 + 11e^{-900(0.0003205297)(10)}}$$

$$y \approx 557.432$$

So almost 558 students have contracted pinkeye in time to open presents.