Calculus Notes AB

Calculator Procedures

Calculator Settings

Your calculator should be in Radian mode when performing Calculus problems.
(Those of you in Physics need to be in Degree mode while in Physics class. You will be switching modes daily).

Graphing Windows

Push ZOOM for pre-set windows

1) ZDecimal is the best for tracing (-4.7 to 4.7). (-6.3 to 6.3 for TI-89)
2) ZStandard shows most of the graph (-10 to 10).
3) ZoomFit will help you find the graph if the graph is not in your current window.

Intersection

TI - 83 / 84

Push CALC Intersection

1) Move cursor near point of intersection.
2) Push enter for first equation.
3) Push enter for second equation.
4) Push enter for guess.

TI - 89

Push CALC Intersection

1) Move cursor near point of intersection.
2) Push enter for first equation.
3) Push enter for second equation.
4) Move cursor to left of intersection and press enter.
5) Move cursor to right of intersection and press enter.

Zeroes

Push CALC Zero

1) Move cursor to the left of intercept (lower bound) and push enter.
2) Move cursor to the right of intercept (lower bound) and push enter.
3) Push enter for guess

Derivatives

	Derivative at all points	2nd Derivative	Derivative at a point
TI-83:	NA	NA	$nDeriv(\text{equation}, x, \text{point})$
TI-84:	NA	NA	$\frac{d}{dx}(\text{equation})\vert x = \text{point}$
TI-89:	$d(equation, x)$	$d(equation, x, 2)$	$d(\text{equation}, x)\vert x = \text{point}$

Integrals

	Indefinite Integral	Definite Integral
TI-83:	NA	$fnInt(equation,\ x,\ lower\ bound,\ upper\ bound)$
TI-84:	NA	$\int_{lower\ bound}^{upper\ bound}(equation)\,dx$
TI-89:	$\int(equation,\ x)$	$\int(equation,\ x,\ lower\ bound,\ upper\ bound)$

Pre-Calculus Review

BOWTIE METHOD

Bowtie Method is a shortcut method to add and subtract fractions.

EX: $\frac{2}{3}+\frac{5}{7}=\frac{7\cdot 2+3\cdot 5}{3\cdot 7}=\frac{29}{21}$

EX: $\frac{5}{x}-\frac{x}{4}=\frac{4\cdot 5-x\cdot x}{x\cdot 4}=\frac{20-x^2}{4x}$

EX#1: $\frac{2}{3}+\frac{5}{7}=$

EX#2: $\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=$

EX#3: $\frac{3}{x}+\frac{6}{x+4}=$

INEQUALITIES

EX#1: Absolute Value Inequalities

$a)$ $|x-1|>3$

$b)$ $|-2x+3|\le 2$

$c)$ $|3-x|\ge -4$

$d)$ $|2x-5|<-3$

EX#2:

$a)$ $3x^2+18x\ge 0$

$b)$ $x^2+3x-10\le 0$

$c)$ $\frac{(x+1)(x-5)}{(x-2)}>0$

$d)$ $\frac{1}{x}<3$

INFORMATION ABOUT LINES:

To **write an equation** of a line **you need:** **1) Slope and 2) A point**

Slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

**** **Point - slope form:** $y - y_1 = m(x - x_1)$ (BEST FORM)

** **Slope - intercept form:** $y = mx + b$ (m = slope and b = y-intercept)

Standard form: $Ax + By = C$ (A must be positive and A, B, and C must be integers)

General form: $Ax + By + C = 0$ (A must be positive and A, B, and C must be integers)

Parallel lines have the **same** slope.

Perpendicular lines have **negative reciprocal** slopes of each other. $\left(\text{If } m = 5, \text{ then } m_\perp = \frac{-1}{5}\right)$

WRITING EQUATIONS OF LINES

EX: Write the equation of the line that passes through $(2,-3)$ and $(5,7)$.

$$m = \frac{7-(-3)}{5-2} = \frac{10}{3}$$

Point - slope form: $y + 3 = \frac{10}{3}(x-2)$ or $y - 7 = \frac{10}{3}(x-5)$ You can use either point.

Slope - intercept form: $y = \frac{10}{3}x - \frac{29}{3}$

Standard form: $10x - 3y = 29$ **General form:** $10x - 3y - 29 = 0$

EX#1: Write the equation of each line

$a)$ $(4,-7)$; $m = 11$

$b)$ $(-5,-8)$ $(11,-10)$

EX#2: Write the equation of the line that is....

$a)$ parallel to the line $y = -3x + 8$ and passes thru the point $(-6,4)$.

$b)$ $\perp$ to the line $4x - 5y = 7$ and passes thru the point $(9,-1)$.

EVEN AND ODD FUNCTIONS

A function is **even** iff it is **symmetric about the y - axis.**

If $f(x)=f(-x)$ then the graph is symmetric about the y-axis. (Even Function)

EX : $f(x)=x^4+4x^2$ and $g(x)=7-x^2$ are both even functions because all the powers are even.

y = cos x is an even function because it is symmetric about the *y - axis*.

A function is **odd** iff it is **symmetric about the origin.**

If $f(x)=-f(-x)$ then the graph is symmetric about the origin. (Odd Function)

EX : $f(x)=x^3+5x^5$ and $g(x)=7x-x^3$ are both odd functions because all the powers are odd.

y = sin x is an odd function because it is symmetric about the *origin*.

A function is neither **even** or **odd** if its' powers are mixed (some even and some odd).

EX : $f(x)=x^2+5x^5$ and $g(x)=7-x^3$ are neither even or odd functions because the powers are mixed.

COMPOSITION OF FUNCTIONS : $(f\circ g)(x)$ and $(g\circ f)(x)$

$(f\circ g)(x)=f(g(x))$ means plug $g(x)$ into $f(x)$

$(g\circ f)(x)=g(f(x))$ means plug $f(x)$ into $g(x)$

$(f\circ g)(2)=f(g(2))$ which means plug 2 into $g(x)$ and that answer into $f(x)$.

$(g\circ f)(6)=g(f(6))$ which means plug 6 into $f(x)$ and that answer into $g(x)$.

EX #1 : Given $f(x)=x^2-5$ and $g(x)=3x+6$

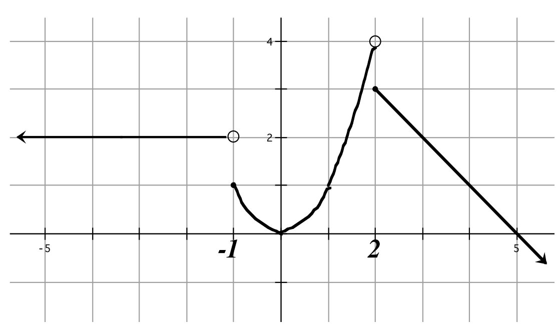
$(f\circ g)(x)=(3x+6)^2-5 \;=\; 9x^2+36x+31$ $\qquad$ $(f\circ g)(1)=f(g(1))=f(9)=76$

$(g\circ f)(x)=$ $\qquad$ $(g\circ f)(8)=$

$(g\circ g)(x)=$ $\qquad$ $(f\circ f)(-2)=$

PIECE - WISE FUNCTIONS :

EX : Graph $f(x)=\begin{cases} 2 & for \quad x<-1 \\ x^2 & for \quad -1\le x<2 \\ 5-x & for \quad x\ge 2 \end{cases}$



DOMAIN

DOMAIN : the x values that work in an equation. (How far left and right the graph goes)

There are three things in math that you cannot do and they all will restrict your domain

1) You cannot divide by zero.
2) You cannot take the square root of a negative number.
3) You cannot take the logarithm of a negative # or zero.

If there are no fractions, radicals, or logarithms, then the domain has no restrictions and therefore the domain is all real #'s.

For fractions the denominator cannot = 0, so the # that makes the bottom = 0 is not in the domain.

EX : $f(x)=\dfrac{3}{x-11}$

Domain : $x \neq 11$

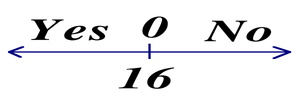
EX : $f(x)=\dfrac{x-8}{x^2+3x-10}=\dfrac{x-8}{(x+5)(x-2)}$

Domain : $x \neq -5, 2$

Since you cannot take the square root of a negative # the function inside the $\sqrt{\ }$ must be ≥ 0

EX : $f(x)=\sqrt{16-x}$

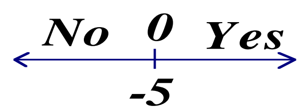
$$16-x \geq 0$$
$$16 \geq x$$



Domain : $x \leq 16$

EX : $f(x)=\sqrt{x+5}$

$$x+5 \geq 0$$
$$x \geq -5$$



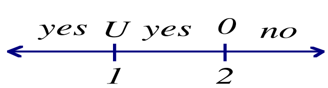
Domain : $x \geq -5$

Since you cannot take the square root of a negative # or divide by zero, you must find #'s that make the square root ≥ 0 but don't make the denominator $= 0$. (The best technique is to organize yourself with a number line)

EX : $f(x)=\dfrac{\sqrt{2-x}}{x-1}$

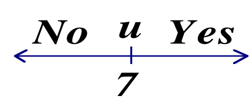
Domain : $x \leq 2,\ x \neq 1$

or $(-\infty,\ 1)(1,\ 2]$



EX : $f(x)=\dfrac{10}{\sqrt{x-7}}$

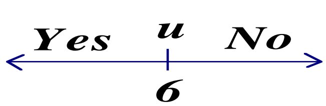
Domain : $x > 7$



Since we cannot take the log/ ln of a negative # or zero, the function that we are taking the log/ln of must be >0.

EX : $f(x)=\ln(6-x)$

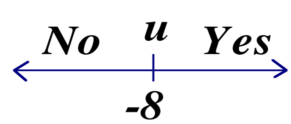
$$6-x>0$$
$$6>x$$



Domain : $x<6$

EX : $f(x)=\log(x+8)$

$$x+8>0$$
$$x>-8$$



Domain : $x>-8$

If there are no fractions, radicals or logarithms, the function has no restrictions and has a domain of all real #'s.

EX : $f(x)=x^3$

Domain : all real #'s

EX : $f(x)=\dfrac{8}{x^2+5}$

Domain : all real #'s

Domain Practice

EX#1: $f(x)=\dfrac{x-2}{40+x}$

EX#2: $f(x)=\dfrac{x}{x^2-36}$

EX#3: $f(x)=\dfrac{8}{x^2+5}$

EX#4: $f(x)=\dfrac{\sqrt{x+9}}{x-4}$

EX#5: $f(x)=\sqrt{x-7}$

EX#6: $f(x)=\dfrac{x}{\sqrt{x-17}}$

EX#7: $f(x)=\ln(9+x)$

EX#8: $f(x)=\log(x-3)$

EX#9: $f(x)=\ln\left(x^2+8\right)$

EX#10: $f(x)=\sqrt{x^2-25}$

EX#11: $f(x)=\sin x$

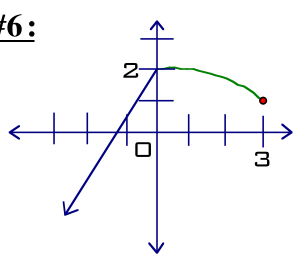
EX#12: $f(x)=\dfrac{\sqrt{x-20}}{x-10}$

RANGE: the y values that work in an equation. (How high and low the graph goes)

INCREASING: as x gets larger the y values get larger. (The graph goes up to the right)

DECREASING: as x gets larger the y values get smaller. (The graph goes down to the right)

EX#6:



Domain:

Interval increasing:

Range:

Interval decreasing:

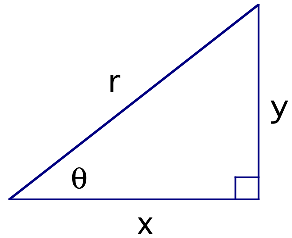
MEMORIZE

TRIGONOMETRIC FUNCTIONS

$$\sin x = \frac{opposite}{hypotenuse} \qquad \csc x = \frac{hypotenuse}{opposite}$$

$$\cos x = \frac{adjacent}{hypotenuse} \qquad \sec x = \frac{hypotenuse}{adjacent}$$

$$\tan x = \frac{opposite}{adjacent} \qquad \cot x = \frac{adjacent}{opposite}$$

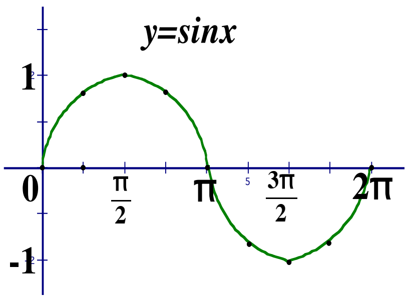


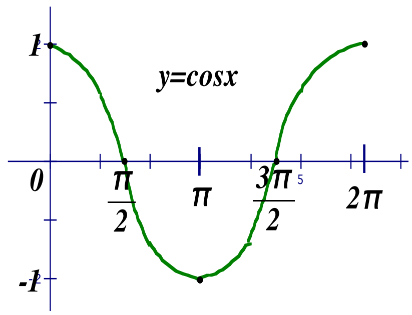
TRIGONOMETRIC FUNCTIONS

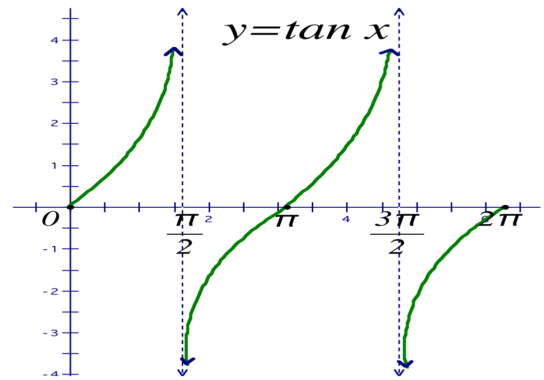
$$\sin x = \frac{y}{r} \qquad \csc x = \frac{r}{y}$$

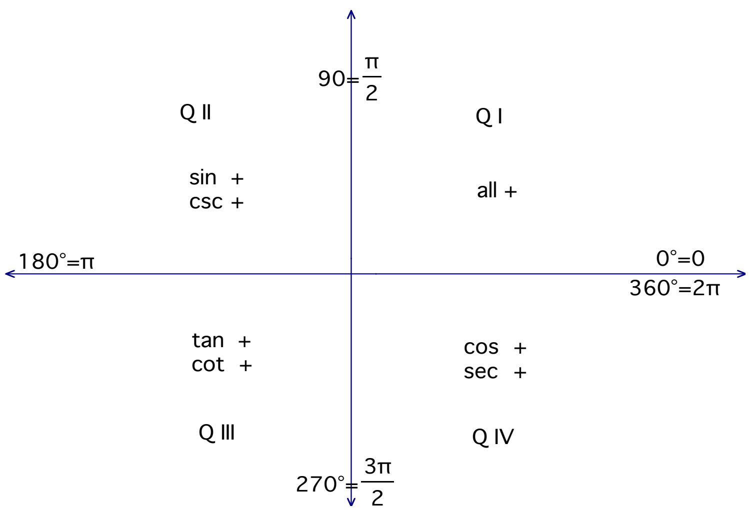
$$\cos x = \frac{x}{r} \qquad \sec x = \frac{r}{y}$$

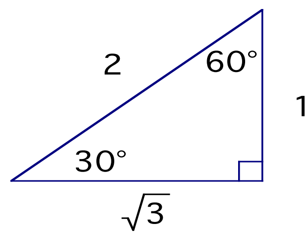
$$\tan x = \frac{y}{x} \qquad \cot x = \frac{x}{y}$$

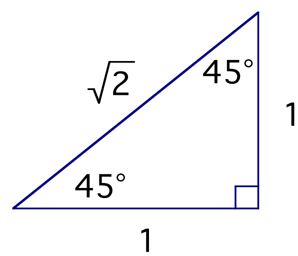












Converting Angles

Degrees to Radians

multiply by $\frac{\pi}{180}$

Radians to Degrees

multiply by $\frac{180}{\pi}$

EX: Convert from degree to radian

$a)\quad 30^\circ \cdot \frac{\pi}{180} = \frac{\pi}{6}$

$b)\quad 45^\circ \cdot \frac{\pi}{180} = \frac{\pi}{4}$

$c)\quad 60^\circ \cdot \frac{\pi}{180} = \frac{\pi}{3}$

$d)\quad 90^\circ \cdot \frac{\pi}{180} = \frac{\pi}{2}$

$e)\quad 120^\circ \cdot \frac{\pi}{180} = \frac{2\pi}{3}$

$f)\quad 135^\circ \cdot \frac{\pi}{180} = \frac{3\pi}{4}$

$g)\quad 150^\circ \cdot \frac{\pi}{180} = \frac{5\pi}{6}$

$h)\quad 180^\circ \cdot \frac{\pi}{180} = \pi$

EX: Convert from radian to degree

$a)\quad \frac{7\pi}{6} \cdot \frac{180}{\pi} = 210^\circ$

$b)\quad \frac{5\pi}{4} \cdot \frac{180}{\pi} = 225^\circ$

$c)\quad \frac{4\pi}{3} \cdot \frac{180}{\pi} = 240^\circ$

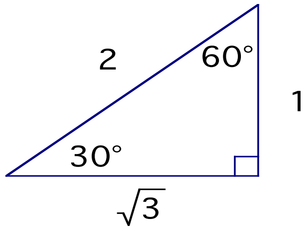
$d)\quad \frac{3\pi}{2} \cdot \frac{180}{\pi} = 270^\circ$

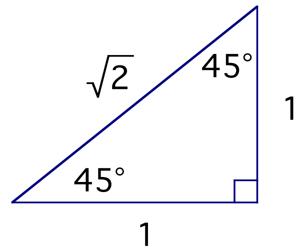
$e)\quad \frac{5\pi}{3} \cdot \frac{180}{\pi} = 300^\circ$

$f)\quad \frac{7\pi}{4} \cdot \frac{180}{\pi} = 315^\circ$

$g)\quad \frac{11\pi}{6} \cdot \frac{180}{\pi} = 330^\circ$

$h)\quad 2\pi \cdot \frac{180}{\pi} = 360^\circ$

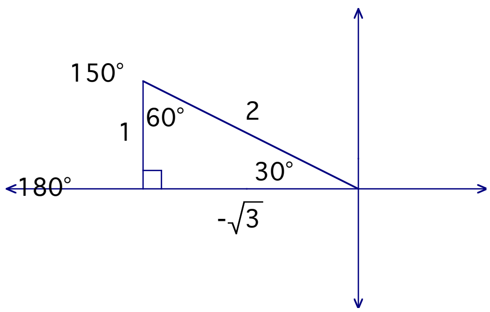




Use triangles above to solve each.

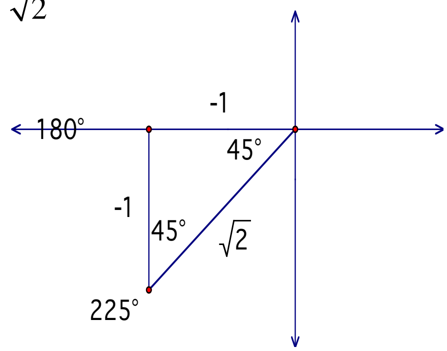
EX :

$\sin 150° = \frac{1}{2}$ $\cos 150° = \frac{-\sqrt{3}}{2}$ $\tan 150° = \frac{-1}{\sqrt{3}}$



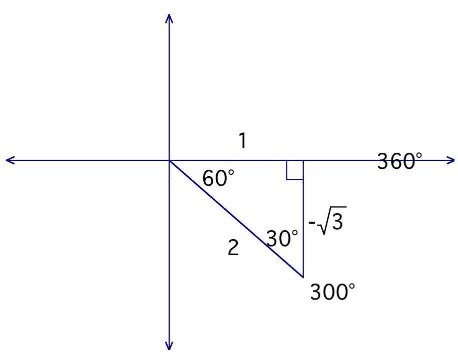
EX :

$\cos 225° = \frac{-1}{\sqrt{2}}$ $\csc 225° = -\sqrt{2}$ $\cot 225° = -1$



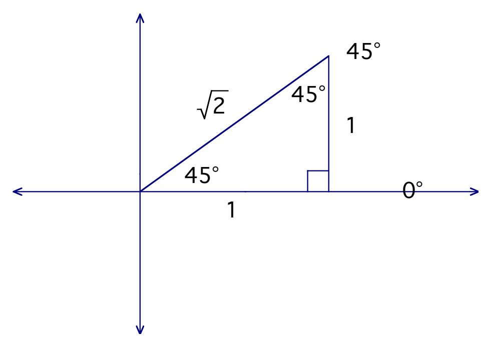
EX :

$\sin 300° = \frac{-\sqrt{3}}{2}$ $\sec 300° = 2$ $\tan 300° = -\sqrt{3}$



EX :

$\csc 45° = \sqrt{2}$ $\sec 45° = \sqrt{2}$ $\cot 45° = 1$



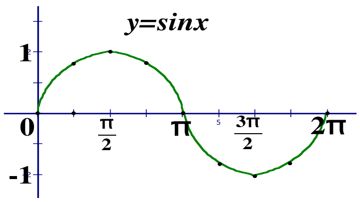
EX #1 :

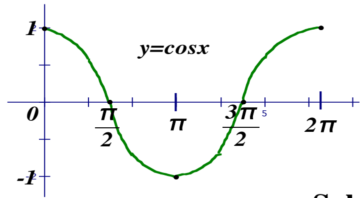
$\sin 120° =$ $\sec 120° =$ $\tan 120° =$

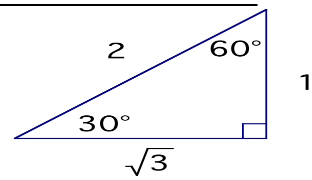
EX#2 :

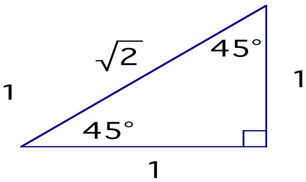
$\csc 210° =$ $\cos 210° =$ $\cot 210° =$

Inverse Trigonometric Functions



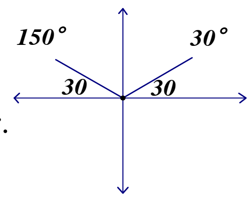






Solve each from $[0°, 360°]$

EX: $\sin\theta = \frac{1}{2}$



Use triangles to find first answer.

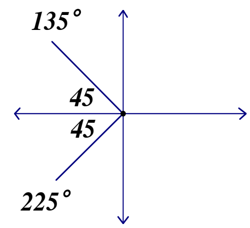
$\theta = \sin^{-1}\frac{1}{2} \quad \Rightarrow \quad \theta = 30°$

To find second answer look in proper quadrant. Sine is positive in Q1 and Q2. Second answer is 30° (reference angle) above x-axis in Q2 which is 150°.

$\theta = 30°, 150°$

Solve each from $[0°, 360°]$

EX: $\cos\theta = \frac{-1}{\sqrt{2}}$



Use triangles to find first answer.

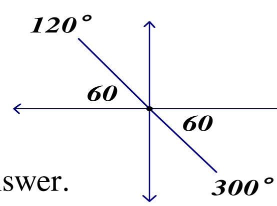
$\theta = \cos^{-1}\frac{-1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 135°$

To find second answer look in proper quadrant. Cosine is negative in Q2 and Q3. Second answer is 45° (reference angle) below x-axis in Q3 which is 225°.

$\theta = 135°, 225°$

Solve each from $[0°, 360°]$

EX: $\tan\theta = -\sqrt{3}$



Use triangles to find first answer.

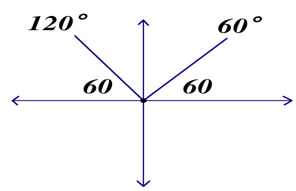
$\theta = \tan^{-1}\left(-\sqrt{3}\right) \quad \Rightarrow \quad \theta = -60° = 300°$

To find second answer look in proper quadrant. Tangent is negative in Q2 and Q4. Second answer is 60° (reference angle) above x-axis in Q2 which is 120°.

$\theta = 120°, 300°$

Solve each from $[0, 2\pi]$

EX: $\csc\theta = \frac{2}{\sqrt{3}}$



Use triangles to find first answer. Must flip first.

$\sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = 60°$

To find second answer look in proper quadrant. Csc is positive in Q1 and Q2. Second answer is 60° (reference angle) above x-axis in Q2 which is 120°.

$\theta = 60°, 120° \qquad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

Solve from $[0°, 360°]$

EX #1: $\sin\theta = \frac{-\sqrt{3}}{2}$

Solve from $[0, 2\pi]$

EX #2: $\tan\theta = \frac{-1}{\sqrt{3}}$

Solve from $[0, 2\pi]$

EX #3: $\cos\theta = \frac{1}{\sqrt{2}}$

Solve from $[0°, 360°]$

EX #4: $\sin\theta = -1$

Solve from $[0, 2\pi]$

EX #5: $\tan\theta = 0$

Solve from $[0, 2\pi]$

EX #6: $\cos\theta = 1$

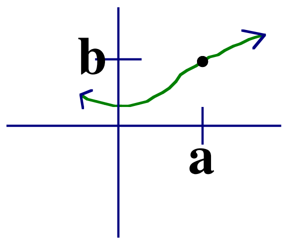
CH.1 LIMITS

** When evaluating limits, we are checking around the point that we are approaching, NOT at the point.

**Every time we find a limit, we need to check from the left and the right hand side
(Only if there is a BREAK at that point).

1-2 Finding Limits Graphically and Numerically

No breaking point

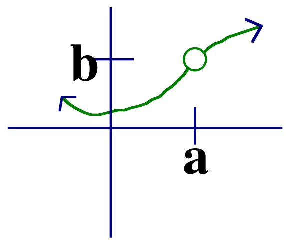


$\lim\limits_{x\to a} f(x) =$

$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

Hole in the graph

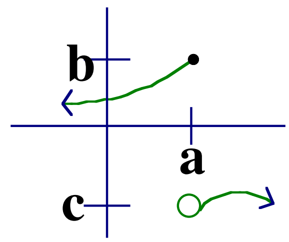


$\lim\limits_{x\to a} f(x) =$

$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

piece-wise function

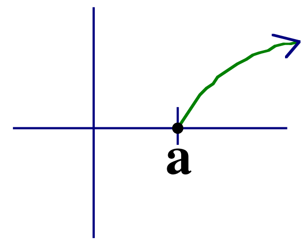


$\lim\limits_{x\to a} f(x) =$

$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

radicals

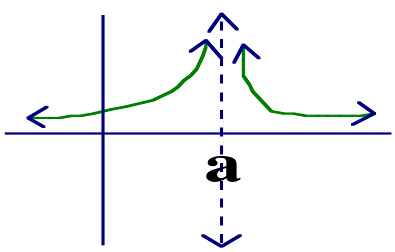


$\lim\limits_{x\to a} f(x) =$

$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

Asymptotes

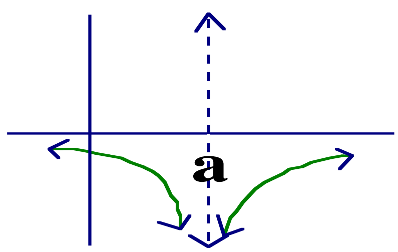


$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

$\lim\limits_{x\to a} f(x) =$

Asymptotes

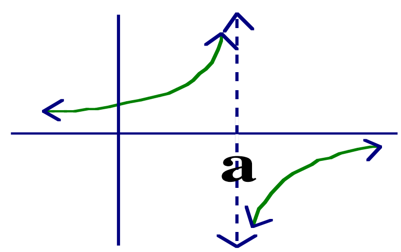


$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

$\lim\limits_{x\to a} f(x) =$

Asymptotes



$\lim\limits_{x\to a^+} f(x) =$

$\lim\limits_{x\to a^-} f(x) =$

$\lim\limits_{x\to a} f(x) =$

**If left and right hand limits DISAGREE, then the limit Does Not Exist (DNE) at that point.

**If left and right hand limits AGREE, then the limit exists at that point as that value.

**Even if you can plug in the value, the limit might not exist at that point. It might not exist from the left or right side or the two sides will not agree.

$\lim\limits_{x\to a^+} f(x) =$ right hand limit $\qquad$ $\lim\limits_{x\to a^-} f(x) =$ left hand limit

**Breaking Points are points on the graph that are undefined or where the graph is split into pieces.

Breaking Points:

1) Holes (when the numerator and denominator equals 0)
2) Radicals (when the radical equals 0)
3) Asymptotes (when the denominator equals 0)
4) Piece-wise functions (the # where the graph is split)

Note: In general when doing limits, $\frac{\#}{0} = \infty$ $\quad \frac{-\#}{0} = -\infty$ $\quad \frac{\#}{\infty} = 0$

**** In limits, if the two sides of a graph don't agree, then the limit does not exist.**

1-3 Analyzing Limits Analytically

LIMITS AT NON-BREAKING POINTS (Very easy. Just plug in the #)

EX#1: $\lim_{x \to 5} x^2 =$ **EX#2:** $\lim_{x \to 9} \sqrt{x-4} =$ **EX#3:** $\lim_{x \to -2} \frac{x-1}{x+1} =$

HOLES IN THE GRAPH $\left(\frac{0}{0}\right)$ (Factor and cancel or multiply by the conjugate and cancel, then plug in #)

EX#1: $\lim_{x \to 2} \frac{x^2+x-6}{x-2} =$

EX#3: $\lim_{x \to -2} \frac{\sqrt{x+11}-3}{x+2} =$

EX#2: $\lim_{x \to 3} \frac{x-3}{x-3} =$

EX#4: $\lim_{x \to 25} \frac{\sqrt{x}-5}{x-25} =$

TRIG. FUNCTIONS

Trig. Identities to know: $\sin 2x = 2\sin x \cos x$ $\quad \cos 2x = \cos^2 x - \sin^2 x$

FACTS: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\quad \lim_{x \to 0} \frac{1-\cos x}{x} = 0$ $\quad \lim_{x \to 0} \frac{\tan x}{x} = 1$

$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$ $\quad \lim_{x \to 0} \frac{1-\cos ax}{bx} = 0$ $\quad \lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

EX#1: $\lim_{x \to 0} \frac{\sin 2x}{11x} =$

EX#2: $\lim_{x \to 0} \frac{2\tan 3x}{7x} =$

EX#3: $\lim_{x \to \pi/2} \frac{6\sin x}{x} =$

EX#4: $\lim_{x \to 0} \frac{\sin x \tan x}{x^2} =$

EX#5: $\lim_{x \to 0} \frac{6\sin x \cos x}{5x} =$

EX#6: $\lim_{x \to 0} \frac{2\sin^2 x \tan^2 x}{x^4} =$

1-4 Continuity and One-Sided Limits

PIECE-WISE FUNCTIONS

$$f(x)=\begin{cases} -1-4x & x<-1 \\ x^2+2 & -1\le x<3 \\ 20 & x\ge 3 \end{cases}$$

The breaking points are -1 and 3.

EX #1: $\lim\limits_{x\to -1^+} f(x)=$

EX #2: $\lim\limits_{x\to 3^+} f(x)=$

EX #3: $\lim\limits_{x\to -1^-} f(x)=$

EX #4: $\lim\limits_{x\to 3^-} f(x)=$

EX #5: $\lim\limits_{x\to -1} f(x)=$

EX #6: $\lim\limits_{x\to 3} f(x)=$

EX #7: $\lim\limits_{x\to 6^+} f(x)=$

EX #8: $\lim\limits_{x\to -4} f(x)=$

EX #9: $\lim\limits_{x\to 2} f(x)=$

$$f(x)=\begin{cases} x^3 & x<5 \\ 30-6x & x>5 \end{cases}$$

EX #1: $\lim\limits_{x\to 5^+} f(x)=$

EX #2: $\lim\limits_{x\to 5^-} f(x)=$

EX #3: $\lim\limits_{x\to 5} f(x)=$

CONTINUITY

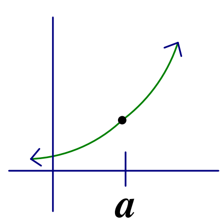
Continuous functions have **no breaks in them**.

Discontinuous functions have **breaks in them (Asymptotes or Holes / Open Circles).**

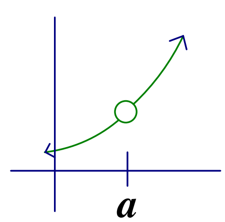
**To check for continuity at "a", you must check left hand limits $\lim\limits_{x\to a^-} f(x)$, right hand limits $\lim\limits_{x\to a^+} f(x)$ and the value of the function at that point $f(a)$. If all three are equal then the function is continuous at a.

If $f(a)=\lim\limits_{x\to -a^-} f(x)=\lim\limits_{x\to -a^+} f(x)$ then the function is **continuous at a**.

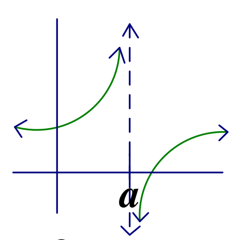
If **$f(a)$ is not equal to either one-sided limit**, then **the function is not continuous (discontinuous) at a.**



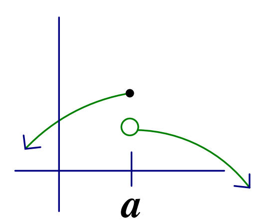
Continuous at a



Point Discontinuity



Infinite Discontinuity



Jump Discontinuity

1-5 Infinite Limits

ASYMPTOTES $\left(\#/0\right)$ (Since the point DNE we have to check a point that is close on the side we are approaching)

There are three possible answers when checking near the breaking point (the # that makes bottom = zero)

1) ∞ $\rightarrow$ If we get a positive answer the limit approaches ∞

2) $-\infty$ $\rightarrow$ If we get a negative answer the limit approaches $-\infty$

3) DNE $\rightarrow$ If we get a positive answer on one side and a negative answer on the other side, then the limit DNE

EX #1: $\lim_{x\to 3^+} \frac{1}{x-3} =$

EX #2: $\lim_{x\to 3^-} \frac{1}{x-3} =$

EX #3: $\lim_{x\to 3} \frac{1}{x-3} =$

EX #4: $\lim_{x\to 3} \frac{2}{(x-3)^2} =$

EX #5: $\lim_{x\to 8} \frac{-6}{(x-8)^2} =$

EX #6: $\lim_{x\to -5^+} \frac{4x}{x+5} =$

EX #7: $\lim_{x\to 0^-} \frac{\cos x}{x} =$

EX #8: $\lim_{x\to 0} \frac{\cos x}{x} =$

EX #9: $\lim_{x\to \pi/2^-} \tan x =$

3-5 Limits at Infinity

LIMITS THAT APPROACH INFINITY

Check the powers of the numerator and denominator.

1) If the denominator (bottom) is a bigger power the limit = 0.

2) If the numerator (top) is a bigger power the limit = ∞ or $-\infty$.

3) If powers are the same the limit = $\dfrac{\text{coefficient of the highest power of numerator}}{\text{coefficient of the highest power of denominator}}$

EX#1: $\lim_{x\to\infty} \frac{2x^2+5}{3x^2-7} =$

EX#2: $\lim_{x\to\infty} \frac{8x}{3x^2+1} =$

EX#3: $\lim_{x\to-\infty} \frac{6x^3}{2x^2-1} =$

EX#4: $\lim_{x\to-\infty} \frac{4x^2}{8-11x^2} =$

EX#5: $\lim_{x\to-\infty} \frac{7x+2}{8x^7-1} =$

EX#6: $\lim_{x\to\infty} \frac{4\cdot 7^x-20}{3\cdot 7^x+2} =$

EX#7: $\lim_{x\to-\infty} \frac{4\cdot 7^x-20}{3\cdot 7^x+2} =$

FINDING VERTICAL ASYMPTOTES AND HOLES

A **vertical asymptote** is the # that makes only the denominator = 0.

A **hole** occurs at the points that make the numerator and denominator = 0 at the same time.

EX#1:

$a)$ $f(x)=\frac{x+2}{x-3}$

vert.asym. hole

$b)$ $f(x)=\frac{x}{x^2+7}$

vert.asym. hole

$c)$ $f(x)=\frac{(x+5)(x-3)}{(x+5)}$

vert.asym. hole

$d)$ $f(x)=\frac{(x+4)(x+1)}{(x+1)(x-5)}$

vert.asym. holes

CH.2 DIFFERENTIATION

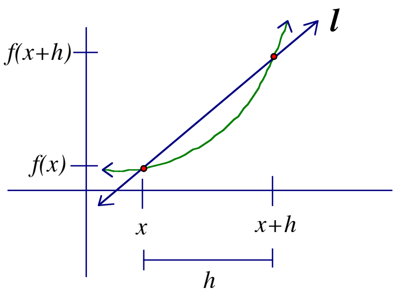
2-1 The Derivative by Definition and the Tangent Line Problem

Derivative at all points (slope at all points)

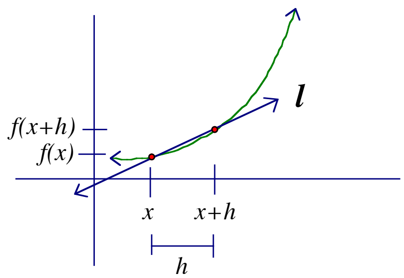
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivative at the point $(a,f(a))$ (slope at a point)

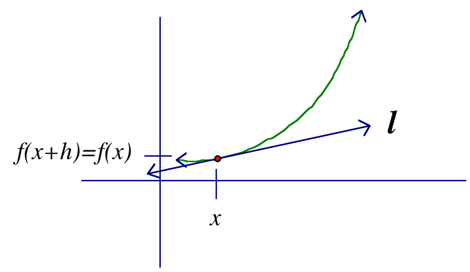
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



Line l is a secant line



Line l is a secant line



Line l is a tangent line

slope of secant line $l=\dfrac{f(x+h)-f(x)}{x+h-x}$

$\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ means that the distance h is approaching 0 and the points get closer to each other and the two points become the same point and line l is now a tangent line.

The derivative of a function finds the slope of the tangent line!

EX #1: $f(x)=x^2$ Find $f'(x)$ Use $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f'(x)=$

EX #2: $f(x)=x^2-3x+2$ Find $f'(x)$ and $f'(4)$. Use $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

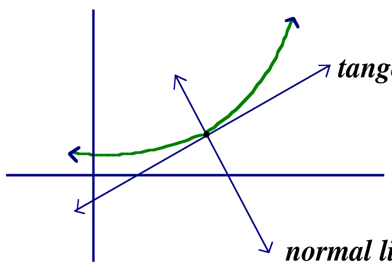
$f'(x)=$

EX #3: $f(x)=x^2-3x+2$ Find $f'(4)$ Use $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

$$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{(4+h)^2-3(4+h)+2-6}{h}=\lim_{h\to 0}\frac{16+8h+h^2-12-3h-4}{h}$$

$$=\lim_{h\to 0}\frac{5h+h^2}{h}=\lim_{h\to 0}5+h=5$$

$\boxed{\text{5 is the slope of the tangent line at } x=4.}$



The **derivative finds the slope of the tangent line**.

The **normal line is perpendicular to the tangent line**.

EX #4: $f(x)=5x^2$ Find equation of the tangent line and normal line at $x=3$.

$f'(3)=$

Equation of the tangent line:

Equation of the normal line:

2-2 Basic Differentiation Rules, Notation and Rates of Change

Properties of Derivatives

Derivative is a **rate of change**; it finds the change in y over the change in x, $\frac{dy}{dx}$, which is slope.

The Derivative finds the slope of the tangent line to a curve!

1st derivative $\Rightarrow$ max. and min., increasing and decreasing, slope of the tangent line to the curve, and velocity.

2nd derivative $\Rightarrow$ inflection points, concavity, and acceleration.

The 2nd Derivative finds the rate of the slopes of the tangent line to a curve.

***Power Rule**

$f(x) = x^n$

$f'(x) = nx^{n-1}$

EX: $y = 4x^3$

$y' = 12x^2$

***Constant Rule**

$f(x) = c$

$f'(x) = 0$

EX: $y = 8$

$y' = 0$

Derivative notation

Lagrange	Leibniz	Newton
$y = x^3$	$y = x^3$	$y = x^3$
$y' = 3x^2$	$\frac{dy}{dx} = 3x^2$	$\dot{y} = 3x^2$
$y'' = 6x$	$\frac{d^2y}{dx^2} = 6x$	$\ddot{y} = 6x$
$y''' = 6$	$\frac{d^3y}{dx^3} = 6$	$\dddot{y} = \dddot{y} = 6$
$y^{(4)} = 0$	$\frac{d^4y}{dx^4} = 0$	$\overset{4}{\dot{y}} = 0$

EX#1: Find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right\|_{x=2}$.

$a)$ $y = 10x^4$

$b)$ What does $\left.\frac{dy}{dx}\right\|_{x=2}$ find?

EX#2: Find $f'(x)$ for each.

$a)$ $f(x) = 100$

$b)$ $f(x) = \frac{1}{x}$

$c)$ $f(x) = \frac{2}{x^2}$

$d)$ $f(x) = \frac{x}{5}$

EX#3: Find the derivative of each

$a)$ $y = \frac{1}{4}x^4 - 8x^3 + 15x + 2$

$b)$ $f(x) = x - \frac{10}{x^7}$

Slope of the tangent line to the curve

EX: Given $f(x)=3x^2-10x$ Find equation of the tangent line and normal line at $x=4$.

$f(x)=3x^2-10x$ $\quad$ $f'(x)=6x-10$

$f(4)=8$ $\quad$ $f'(4)=14$

Equation of a Line (point - slope form): $y-y_1=m(x-x_1)$

Equation of the tangent line: $y-8=14(x-4)$

Equation of the normal line: $y-8=\dfrac{-1}{14}(x-4)$

EX #4: Find the slope, write the equation of the tangent line and the normal line at the given point for each.

$a)$ $f(x)=8x^2-7;\quad x=3$

Equation of the tangent line:

Equation of the normal line:

$b)$ $f(x)=\dfrac{1}{x^2};\quad x=2$

Equation of the tangent line:

Equation of the normal line:

*Trig. Functions

2 STEPS: Derivative of the trig. function $\cdot$ Derivative of the angle

Function	*Derivative*	*Function*	*Derivative*	*Function*	*Derivative*
$\sin x$	$\cos x$	$\tan x$	$\sec^2 x$	$\sec x$	$\sec x\tan x$
$\cos x$	$-\sin x$	$\cot x$	$-\csc^2 x$	$\csc x$	$-\csc x\cot x$

EX #5: Find each derivative

$a)$ $y=\sin(3x)$ $\quad$ $b)$ $y=\cos(x^2)$ $\quad$ $c)$ $y=\tan(3x^3)$

$d)$ $y=\sec(4x^2+1)$ $\quad$ $e)$ $y=\csc(2x^7)$ $\quad$ $f)$ $y=\cot(5x-1)$

$g)$ $y=\sin(\cos x)$ $\quad$ $h)$ $y=\tan(\sec 3x)$

2-3 Product and Quotient Rule

*Product Rule

4 STEPS : Derivative of First equation · Second equation + Derivative of Second equation · First equation

$y = f(x) \cdot g(x)$ $\qquad\qquad \dfrac{dy}{dx} = f'(x)g(x) + g'(x)f(x)$

EX #1 : $f(x) = (2x+3)(x^2-7)$

EX #2 : $f(x) = 2x^3 + 3x^2 - 14x - 21$

*Quotient Rule

5 STEPS : $\dfrac{\textbf{Derivative of Top equation} \cdot \textbf{Bottom equation} - \textbf{Derivative of Bottom equation} \cdot \textbf{Top equation}}{(\textbf{Bottom equation})^2}$

$y = \dfrac{f(x)}{g(x)}$ $\qquad\qquad \dfrac{dy}{dx} = \dfrac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

EX #3 : $f(x) = \dfrac{x^2-1}{x^2+1}$

EX #4 : $f(x) = \dfrac{x}{\cos x}$

EX #5 : Find first three derivatives of $y = x \sin x$

2-4 Chain Rule

We use Chain Rule when the whole problem is to a power.

3 STEPS: 1) Power in front   2) Lower power by 1   3) Multiply by derivative of inside

OR   Derivative of outside function $\cdot$ Derivative of inside function

$y=(f(x))^n$ **OR** $y=f(g(x))$ **OR** $y=f(4x)$ **OR** $y=f(x^2)$

$y'=n(f(x))^{n-1}\cdot f'(x)$ $y'=f'(g(x))\cdot g'(x)$ $y'=f'(4x)\cdot 4$ $y'=f'(x^2)\cdot 2x$

EX: $y=(3x^2-4)^8$

$$\frac{dy}{dx}=8(3x^2-4)^7\cdot 6x \quad \text{(Don't forget to multiply by the derivative of the inside)}$$

$$\frac{dy}{dx}=48x(3x^2-4)^7$$

Find the derivative for each example

EX #1: $a)\quad y=x^5$ $b)\quad y=(x^2+1)^5$

EX #2: $y=4(x^2+2x)^3$

EX #3: $a)\quad y=3x^5$ $b)\quad y=(3x)^5$

EX #4: $f(x)=\sqrt{x^2-1}$

EX #5: $a)\quad y=f(7x^2)$ $b)\quad y=\cos(4x)$ $c)\quad f(x)=\cos^3(4x)$

EX #6: $y=x^2\sqrt{x-3}$

2-5 Implicit Differentiation

The differentiable functions we have encountered so far can be described by equations in which "y" is expressed in terms of "x". We can also find the derivative of the equation expressed in terms of x and y.

***Implicit Differentiation**: function in terms of x's and y's $\left(\text{must write } \dfrac{dy}{dx} \text{ everytime you take a deriv. of } y\right)$

EX: $x^2 - xy + 3y^2 = 7$

$$2x - \left[(1)y + \frac{dy}{dx}x\right] + 6y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 6y\frac{dy}{dx} = -2x + y$$

$$\frac{dy}{dx}(-x + 6y) = -2x + y$$

$$\frac{dy}{dx} = \frac{-2x + y}{-x + 6y}$$

EX#1: Find $\dfrac{dy}{dx}$ for each.

$a)$ $y = x^3$

$b)$ $x = y^3$

EX#2: $x^2 + y^2 = 16$ $(3, \sqrt{7})$

Find the equation of the tangent line at the given point.

EX#3: Find $\dfrac{dy}{dx}$. $5x^2 - xy + y = 12$

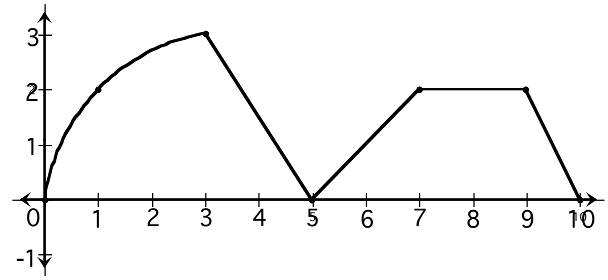
EX #4: Find first two derivatives of $y^2 - x^2 = 20$

Graphs and Charts

Derivative of a graph

$f'(1)=$ $f'(3)=$ $f'(4)=$

$f'(6)=$ $f'(7)=$ $f'(8)=$



Equation of the tangent line at $x=1$

Equation of the tangent line at $x=4$

Derivative of a chart

x	0	15	30	45	60	75	90	105	120
$f(x)$	58	63	72	60	62	69	61	74	67

EX: $f'(30)=\dfrac{60-63}{45-15}=\dfrac{-3}{30}=\dfrac{-1}{10}$

EX: $f'(97.5)=\dfrac{74-61}{105-90}=\dfrac{13}{15}$

EX#1: $f'(75)=$

EX#2: $f'(22.5)=$

How to read Derivative by Definition Problems

This problem means take a derivative of $\sin x$.

EX: $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}=\underline{\cos x}$

This problem means take a derivative of $5x^3$ at $x=2$.

EX: $\lim\limits_{h\to 0}\dfrac{5(2+h)^3-40}{h}=\underline{60}$

Solve each

EX#1: $\lim\limits_{h\to 0}\dfrac{3(x+h)^4-3x^4}{h}=$

EX#2: $\lim\limits_{h\to 0}\dfrac{(1+h)^4-1}{h}=$

EX#3: $\lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan x}{h}=$

EX#4: $\lim\limits_{h\to 0}\dfrac{\cos\left(\frac{\pi}{6}+h\right)-\frac{\sqrt{3}}{2}}{h}=$

2.6 Related Rates

We take derivatives with respect to t which allows us to find velocity. Here is how you take a derivative with respect to t:

derivative of x is $\frac{dx}{dt}$, derivative of y^2 is $2y\frac{dy}{dt}$, derivative of r^3 is $3r^2\frac{dr}{dt}$, derivative of t^2 is $2t\frac{dt}{dt} = 2t$

V means volume ; $\frac{dV}{dt}$ means rate of change of volume (how fast the volume is changing)

r means radius ; $\frac{dr}{dt}$ means rate of change of radius (how fast the radius is changing)

$\frac{dx}{dt}$ is how fast x is changing; $\frac{dy}{dt}$ is how fast y is changing

<u>**Volume of a sphere**</u>	<u>**Surface Area of a sphere**</u>	<u>**Area of a circle**</u>	<u>**Circumference of a circle**</u>
$V = \frac{4}{3}\pi r^3$	$A = 4\pi r^2$	$A = \pi r^2$	$C = 2\pi r$
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$	$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$	$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$	$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

<u>**Unique cases**</u> :

<u>**Volume of a cylinder**</u>

$$V = \pi r^2 h$$

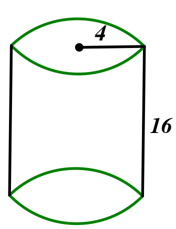
r is not a variable in a cylinder because its' value is always the same

<u>**Volume of a cone**</u>

$$V = \frac{1}{3}\pi r^2 h \quad \text{use } \frac{r}{h} =$$

Due to similar triangles, the ratio of the radius to the height is always the same. Replace r or h depending on what you are looking for.

<u>**EX:**</u> In the cylinder below the radius will always be 4 so we can replace r with 4 BEFORE we take the derivative.
$V = \pi r^2 h$ becomes $V = 16\pi h$.

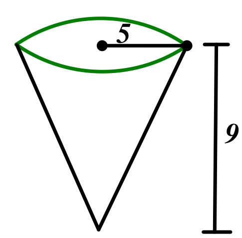


<u>**EX:**</u> In the cone to the right the ratio of the radius and height will always be $\frac{r}{h} = \frac{5}{9}$, so $r = \frac{5}{9}h$ or $h = \frac{9}{5}r$.
We replace the appropriate one

$V = \frac{1}{3}\pi r^2 h$ becomes:

$$V = \frac{1}{3}\pi\left(\frac{5}{9}h\right)^2 h \Rightarrow V = \frac{25}{243}\pi h^3$$

<u>OR</u> $V = \frac{1}{3}\pi r^2\left(\frac{9}{5}r\right) \Rightarrow V = \frac{3}{5}\pi r^3$

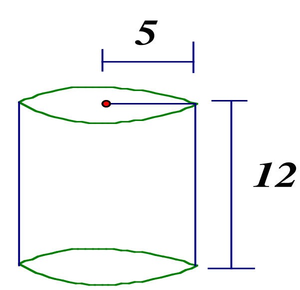


EX #1: Given $2x^2 - 5y^2 = 27$ find $\dfrac{dy}{dt}$, when $x = 6,\ y = -3$ and $\dfrac{dx}{dt} = 4$.

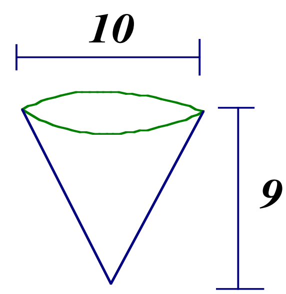
EX #2: Water is spilling on the ground. The radius is changing at 3 in/s. How fast is the area changing when the radius is 8 in. ? $(A = \pi r^2)$

EX #3: Suppose a spherical balloon is inflated at the rate of 10 in^3/min. How fast is the radius of the balloon changing when the radius is 5 inches? $\left(V = \dfrac{4}{3}\pi r^3\right)$

EX #4: Water is poured into a cylinder with dimensions below at the rate of 13 in^3/s. How fast is the height of the water changing when the height is 4 inches? $\left(V = \pi r^2 h\right)$

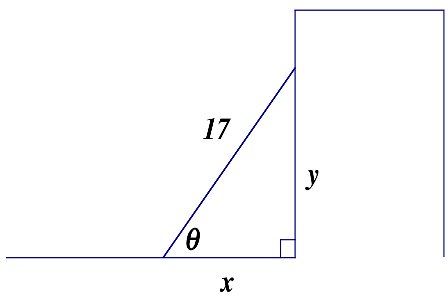


EX #5 : Water is leaking out of a cone with diameter 10 inches and height 9 inches at the rate of 7 in^3/s. How fast is the radius of the water changing when the radius is 3 inches ?



EX #6 : A 17 foot ladder is leaning against the wall of a house. The base of the ladder is pulled away at 2 ft. per second.

a) How fast is the ladder sliding down the wall when the base of the ladder is 8 ft. from the wall?

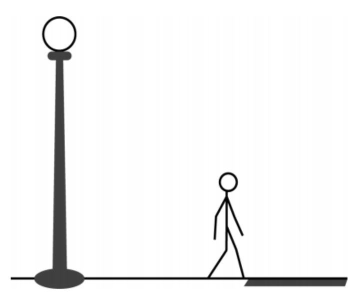


b) How fast is the area of the triangle formed changing at this time?

c) How fast is the angle between the bottom of the ladder and the floor changing at this time?

EX #7 : A person 6 ft. tall walks directly away from a streetlight that is 13 feet above the ground. The person is walking away from the light at a constant rate of 3 feet per second.

a) At what rate, in feet per second, is the length of the shadow changing?



b) At what rate, in feet per second, is the tip of the shadow changing?

CH.3 APPLICATIONS OF DIFFERENTIATION

3-1 Extrema on an Interval

Let f be continuous on a closed bounded interval $[a,b]$. Then f has an absolute maximum and absolute minimum on the interval $[a,b]$.

Procedure for finding absolute max. and absolute min.:

Compute the values of f at all critical points (when $f'(x)=0$) in (a,b) and at the endpoints a and b.

The largest of these values is the absolute maximum value of f on $[a,b]$.

The smallest of these values is the absolute minimum value of f on $[a,b]$.

Find the extreme values (max. and min.) of f on given interval and determine at which #'s in they occur.

EX#1: Let $f(x)=x^2-8x+15$; $[-1\ ,7]$

EX#2: Let $f(x)=6x-x^3$; $[0,4]$

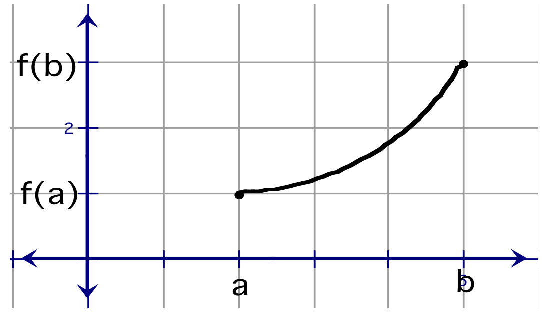
There are critical points that show up as undefined in the derivative, but work in the original equation. I call these special points **Hardpoints**. The textbooks don't have a name for these points so I gave them one.

Find the extreme values (max. and min.) of f on given interval and determine at which #'s in they occur.

EX#3: $f(x)=x^{2/3}-3$; $[-1, 8]$

EX#4: $f(x)=|x-3|$; $[1, 10]$

<u>3-2 Mean Value Theorem and Rolle's Theorem</u>



<u>*Mean - Value Theorem</u>

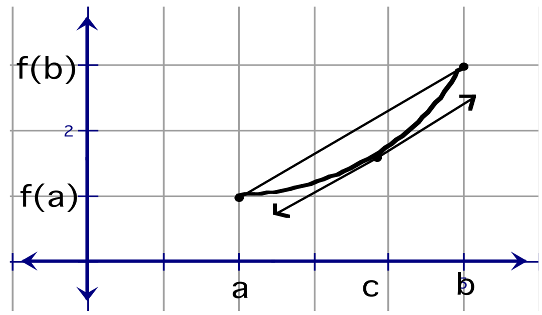
(Only <u>**applies**</u> if the <u>**function is continuous and differentiable**</u>)

<u>Slope of tangent line</u> = <u>slope of line between two points</u>

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

According to the Mean Value Theorem, <u>**there must be a number c**</u> between a and b that the slope of the tangent line at c is the same as the slope between points $(a, f(a))$ and $(b, f(b))$.

The <u>slope of secant line from a and b</u> is the <u>same</u> as <u>slope of tangent line through c</u>.



Use the Mean Value Theorem to find all values of c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

<u>**EX #1:**</u> $f(x)=\sin x \quad (0,\pi)$

<u>**EX #2:**</u> $f(x)=-x^2+5 \quad (-1,2)$

<u>**EX #3:**</u> $f(x)=|x-2| \quad (0,5)$

3-3 Increasing and Decreasing Functions and the First Derivative Test

Properties of First Derivative

increasing : slopes of tangent lines are positive (derivative is positive.) $f'(x) > 0$

decreasing : slopes of tangent lines are negative (derivative is negative). $f'(x) < 0$

maximum point : Slopes switch from positive to negative at maximum point. $\left(\text{found by setting } f'(x) = 0\right)$

minimum point : Slopes switch from negative to positive at minimum point. $\left(\text{found by setting } f'(x) = 0\right)$

***1st Derivative Test

$a)$ If f' changes from **positive to negative** at the critical point c, then f has a **relative maximum value** at c.

$b)$ If f' changes from **negative to positive** at the critical point c, then f has a **relative minimum value** at c.

Find relative extreme values and determine the intervals on which $f(x)$ is increasing and decreasing.

EX #1 : $f(x) = 4x^3 + 9x^2 - 12x + 3$

EX #2 : $f(x) = \dfrac{x^2}{x-4}$

EX #3 : $f(x) = \sin x \quad [0, 2\pi]$

3-4 Concavity and the Second Derivative Test

Properties of Second Derivative

concave up: slopes of tangent lines are increasing (2nd Derivative is positive). $f''(x) > 0$

concave down: slopes of tangent lines are decreasing (2nd Derivative is negative). $f''(x) < 0$

inflection points: points where the graph switches concavity. (found by setting $f''(x) = 0$)
slopes of tangent line switch from increasing to decreasing or vice versa.

EX #1: $f(x) = 3x^4 - 4x^3$

Find all inflection points and the intervals on which $f(x)$ is concave up and concave down.

EX #2: $f(x) = \cos x \quad [0, 2\pi]$

Find all inflection points and the intervals on which $f(x)$ is concave up and concave down.

*****2nd Derivative Test** (Alternate method for finding rel. max. and rel. min.)

Set $f'(c) = 0$ to find your critical points c. Plug your critical point(s) into $f''(x)$.

$a)$ If $f''(c) < 0$, then c is a rel. max.

$b)$ If $f''(c) > 0$, then c is a rel. min.

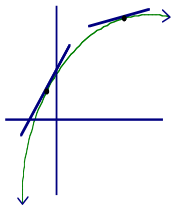
c) If $f''(c) = 0$, then from this test alone we can't draw any conclusions about a relative extreme value at c.

EX #3: Use 2nd Derivative Test to determine if critical points are rel. max. or rel. min.

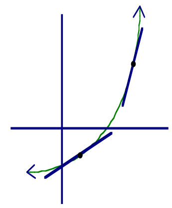
$a)$ $f(x) = x^3 - 48x$

$b)$ $f''(x) = \dfrac{x+6}{x-2}$ and critical points are at $c = 0$ and $c = 5$.

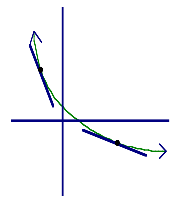
STUDY



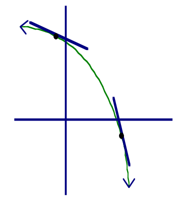
increasing/concave down
slopes are positive and decreasing



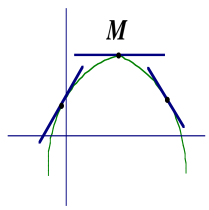
increasing/concave up
slopes are positive and increasing



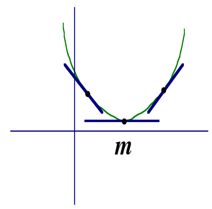
decreasing/concave up
slopes are negative and increasing



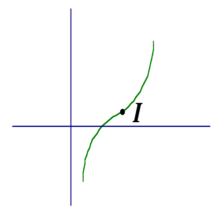
decreasing/concave down
slopes are negative and decreasing



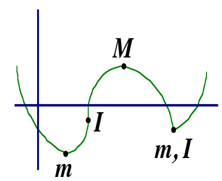
M is a Maximum; slopes switch from positive to negative



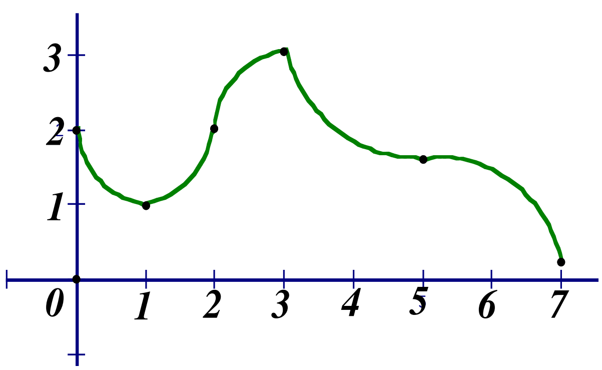
m is a Minimum; slopes switch from negative to positive



I is an inflection point; slopes switch from decreasing to increasing



M is a Maximum; m is a minimum; I is an inflection point



EX #1 : From $[0,7]$ tell me about the function. (Use graph above)

List the x-coordinates for each :	**Find each :**	**On which interval(s) is the graph:**
Inflection points ___________	Abs. max. value_______	increasing/concave up_____________
Relative maximum_________	Abs. min. value________	increasing/concave down___________
Relative minimum_________	Abs. max. value occurs at_______	decreasing/concave up______________
Hard points_______________	Abs. min. value occurs at_______	decreasing/concave down___________

3.6 A Summary of Curve Sketching

GRAPHING TRIG. REVIEW

x - intercepts : where a graph crosses the x-axis. The # that makes $y = 0$.

To find the x-intercept, set the numerator $= 0$ (plug in zero for y).

y - intercepts : where a graph crosses the y-axis. The # found when $x = 0$.

To find the y-intercept, plug in zero for x.

Holes : the # that makes both the numerator and denominator $= 0$.

To find the hole in the graph, you plug the # into the remaining function after canceling out the like factors. Find the hole first before finding the vertical asymptote.

vertical asymptotes : An undefined point on the graph. A graph will never cross the vertical asymptote.

To find the vertical asymptote, set the denominator $= 0$.

The vertical asymptote is $x =$ the # that makes only the denominator $= 0$.

horizontal asymptotes : The graph will approach the horizontal asymptote as x approaches ∞ and $-\infty$.

To find the horizontal asymptote, check the highest powers of the numerator and the denominator.

1) If the denominator (bottom) is a bigger power the horizontal asymptote is $y = 0$.

2) If the numerator (top) is a bigger power there is no horizontal asymptote (there is a different kind of asymptote).

3) If powers are the same the horizontal asymptote is $y = \dfrac{\text{coefficient of the highest power of numerator}}{\text{coefficient of the highest power of denominator}}$

slant asymptotes : Occur when the numerator is one power higher than the denominator.

To find the slant asymptote, you must use long division to divide the denominator into the numerator. The quotient is your slant asymptote.

quadratic asymptotes : Occur when the numerator is two powers higher than the denominator.

To find the quadratic asymptote, you must use long division to divide the denominator into the numerator. The quotient is your quadratic asymptote.

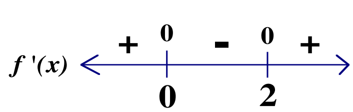
Sketch the graph (Label the maximum, minimum and inflection points)

EX#1 : $y = x^3 - 3x^2$ $\qquad y' = 3x^2 - 6x$ $\qquad y'' = 6x - 6$

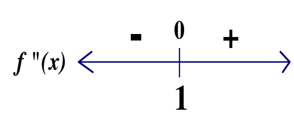
x-int	y-int	v.asym.	h.asym.	rel.max.	rel.min.	inc.	dec.	inf.pts.	conc.up	conc.down
$(0,0)$ $(3,0)$	$(0,0)$	NONE	NONE	$(0,0)$	$(2,-4)$	$(-\infty,0)$ $(2,\infty)$	$(0,2)$	$(1,-2)$	$(1,\infty)$	$(-\infty,1)$

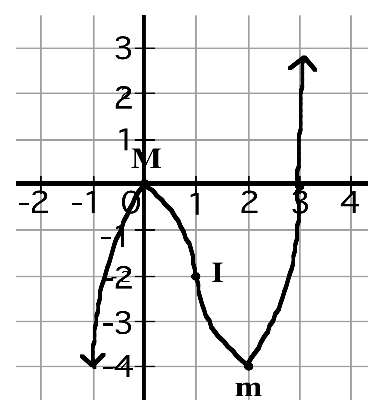
$x^3 - 3x^2 = 0$
$x^2(x-3) = 0$
$x = 0, 3$

$3x^2 - 6x = 0$
$3x(x-2) = 0$
$x = 0, 2$



$6x - 6 = 0$
$6(x-1) = 0$
$x = 1$

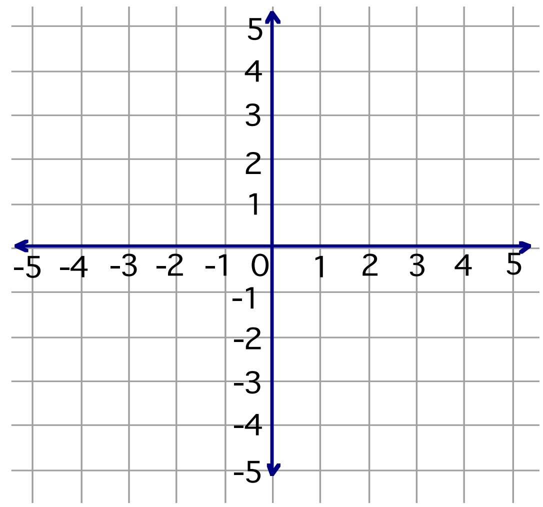




Note all relevant properties of f and sketch the graph (Label the maximum, minimum and inflection points)

EX#2: $y = 3x(x-2)^3$

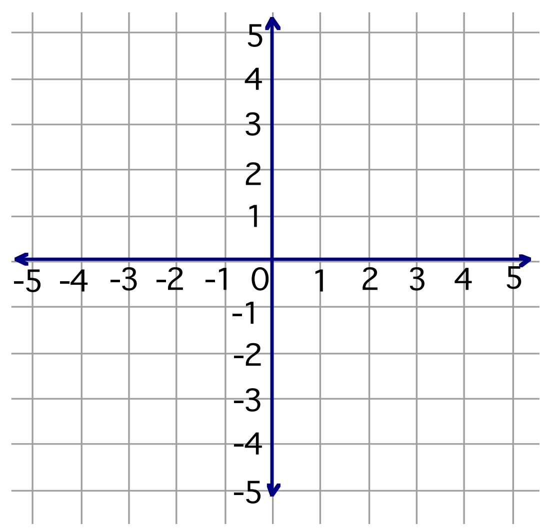
x – int y – int   v.asym.   h.asym.   rel.max.   rel.min.   inc.   dec.   inf .pts.   conc.up   conc.down



Note all relevant properties of f and sketch the graph (Label the maximum, minimum and inflection points)

EX#3: $y = \dfrac{12x}{(x+1)^2}$

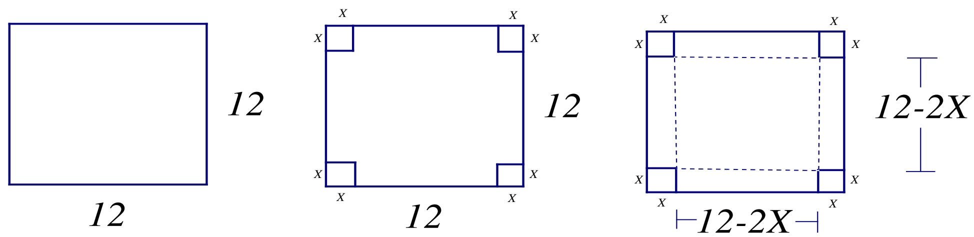
x – int y – int   v.asym.   h.asym.   rel.max.   rel.min.   inc.   dec.   inf .pts.   conc.up   conc.down



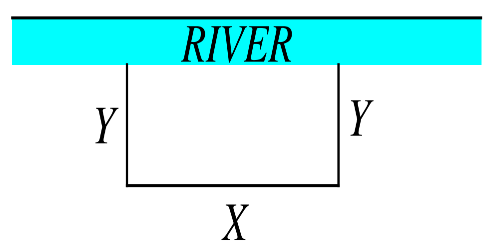
3-7 Optimization Problems

1) Draw and label picture.
2) Write equation based on fact given and write equation for what you need to maximize or minimize.
3) Plug in fact equation into the equation you want to maximize or minimize.
4) Take derivative and set equal to zero.
5) Find remaining information.

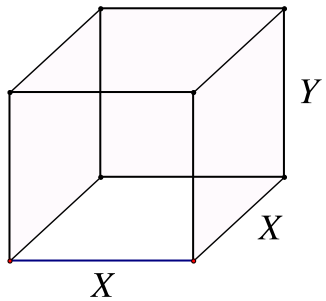
EX#1: An open box of maximum volume is to be made from a square piece of material, 12 inches on a side, by cutting equal squares from the corners and turning up the sides. How much should you cut off from the corners? What is the maximum volume of your box?



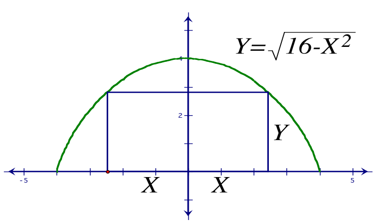
EX#2: A farmer plans to fence a rectangular pasture adjacent to a river. The farmer has 48 feet of fence in which to enclose the pasture. What dimensions should be used so that the enclosed area will be a maximum? What is the maximum area?



EX#3: A crate, open at the top, has vertical sides, a square bottom and a volume of 256 ft^3. What dimensions give us minimum surface area? What is the surface area?



EX#4: A rectangle is bounded by the x-axis and the semicircle $y=\sqrt{16-x^2}$. What length and width should the rectangle have so that its area is a maximum?



3-8 Newton's Method

Newton's method is used to approximate a zero of a function.

*Newton’s Method

$c - \dfrac{f(c)}{f'(c)}$ where c is the current approximation for the zero.

To perform Newton's Method you will need the equation, its' derivative and a first approximation.

EX#1: If Newton’s method is used to approximate the real root of $x^3 + x - 1 = 0$, then a first approximation $x_1 = 1$ would lead to a third approximation of $x_3 =$

$f(x) = x^3 + x - 1$

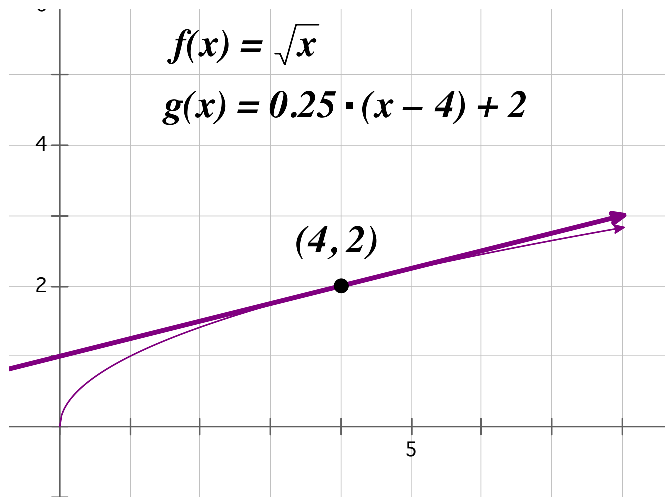
EX#2: Given $f(x) = x^2 - 8$ and first approximation $x_1 = 3$. Find third approximation $x_3 =$.

3-9 Differentials

* Differentials are tangent lines. Tangent lines hug closely along a graph near the point of tangency.
* Sometimes it's easier to use the differential (tangent lines) to approximate a value of a function as opposed to using the function itself.

EX#1: $f(x)=\sqrt{x}$

$a)$ Find the equation of the tangent line (differential) at $(4,2)$.



$b)$ Use the differential to approx. $f(4.2)$.

$c)$ Actual value of $f(4.2)=$

EX#2: $xy^2+3y-x=17$

$a)$ Find the equation of the tangent line at $(1,3)$.

$b)$ Use the differential to approx. $f(1.1)$.

$c)$ Actual value of $f(1.1)=$

CH.4 INTEGRATION

4-1 Antiderivatives and Indefinite Integration

****Integrals:** They find the antiderivatives, sum/total, Area and Volume.

Integration Formulas

$\int (f(x) \pm g(x))dx = \int f(x)\,dx \pm \int g(x)dx$

$\int c \cdot f(x)dx = c \cdot \int f(x)dx$ where c is a constant. Constants can move in and out of an integral.

$\int f(x)dx = F(x) + C$ (notation)

$\int f'(x)dx = f(x) + C$ (notation)

***Integral of a constant**

$\int a\,dx = ax + C$ **EX:** $\int 5\,dx = 5x + C$ **EX:** $\int \pi\,dx = \pi x + C$

***Polynomials**

$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$ **EX:** $\int x^2\,dx = \frac{x^3}{3} + C$ **EX:** $\int 4x^6\,dx = \frac{4x^7}{7} + C$

***Fractions** (Bring up denominator, then take integral)

$\int \frac{1}{x^a}\,dx \Rightarrow \int x^{-a}\,dx = \frac{x^{-a+1}}{-a+1} + C$ **EX:** $\int \frac{1}{x^4}\,dx \Rightarrow \int x^{-4}\,dx = \frac{x^{-3}}{-3} + C = \frac{-1}{3x^3} + C$

***Trig Functions** (**Always divide by derivative of the angle**)

$\int \sin x\,dx = -\cos x + C$ **EX:** $\int \sin 6x\,dx = \frac{-\cos 6x}{6} + C$

$\int \cos x\,dx = \sin x + C$ **EX:** $\int \cos 2x\,dx = \frac{\sin 2x}{2} + C$

EX#1: $a)\ \int x^7\,dx =$ $\qquad b)\ \int 8x^3\,dx =$ $\qquad c)\ \int \sqrt{x}\,dx =$

$d)\ \int 10\,dx =$ $\qquad e)\ \int \frac{9}{x^2}\,dx =$ $\qquad f)\ \int \frac{1}{\sqrt[4]{x^3}}\,dx =$

EX#2: $a)\ \int \left(x^3 - 3x + 1\right)dx =$ $\qquad b)\ \int \frac{x^2+4x}{x}\,dx =$

EX#3: $a)\ \int \cos 9x\,dx =$ $\qquad b)\ \int \sin 4x\,dx =$

Preview of Rectilinear Motion

Symbols used

position: $x(t), y(t), s(t)$

velocity: $v(t)$

acceleration: $a(t)$

Rectilinear Motion

Derivative **position** **velocity** **acceleration** **Integral**

EX#4: $v(t) = t^2 - 4t - 12 \qquad x(0) = 3 \qquad t > 0$

$a)$ Find position at any time t.

$b)$ Find acceleration at any time t.

$c)$ Find velocity at $t = 5$.

$d)$ Find acceleration at $t = 5$.

$e)$ Is the speed inc. or dec. at $t = 5$?

$f)$ When is the particle at rest?

4-3 Riemann Sums and Definite Integrals

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

EX: $\int_2^5 f(x)dx + \int_5^{10} f(x)dx = \int_2^{10} f(x)dx$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

EX: $\int_3^8 f(x)dx = -\int_8^3 f(x)dx$

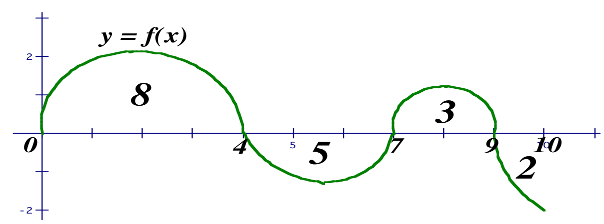
EX #1: Given $\int_1^3 g(x)dx = 8$ $\int_3^6 g(x)dx = 27$ Find the following:

a) $\int_1^6 g(x)dx =$ b) $\int_6^3 g(x)dx =$ c) $\int_3^3 g(x)dx =$ d) $5\int_1^3 g(x)dx =$

EX #2: Given $\int_{-2}^3 f(x)dx = 40$ $\int_3^7 f(x)dx = -17$ $\int_{12}^7 f(x)dx = -10$ Find the following:

a) $\int_{-2}^{12} f(x)dx =$ b) $\int_{12}^3 f(x)dx =$ c) $\int_{-2}^{12} |f(x)|dx =$ d) $\int_{-2}^7 10 \cdot f(x)dx =$

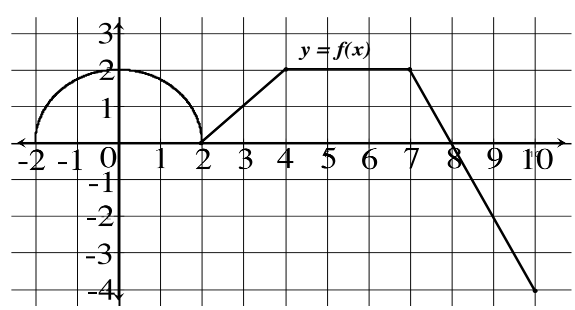
EX #3: **Use picture to the right to find each.**



a) $\int_0^{10} f(x)dx =$ b) $\int_{10}^7 f(x)dx =$

c) $\int_4^{10} |f(x)|dx =$ d) $\int_0^4 (f(x)+10)dx =$

EX #4: **Use picture to the right to find each.**



a) $\int_0^7 f(x)\,dx =$ d) $\int_{-2}^{10} f(x)\,dx =$

b) $\int_4^{-2} f(x)\,dx =$ e) $\int_{-2}^{10} |f(x)|\,dx =$

c) $\int_{10}^4 3 \cdot f(x)\,dx =$ f) $\int_{-2}^{10} (f(x)+5)\,dx =$

4-4 The Fundamental Theorem of Calculus

*** 1st Fundamental Theorem of Calculus

After you take the integral, just plug in the top # minus the bottom #.

$$\int_a^b f(x)\,dx = \left. F(x)\right|_a^b = F(b) - F(a)$$

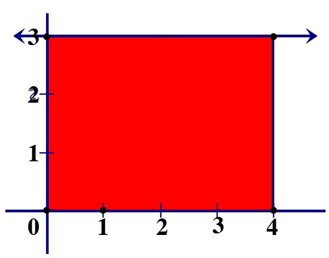
EX #1: Evaluate $\displaystyle\int_0^3 \left(x^2 - 2x - 3\right)dx =$

EX #2: Evaluate $\displaystyle\int_1^9 \frac{1}{\sqrt{x}}\,dx =$

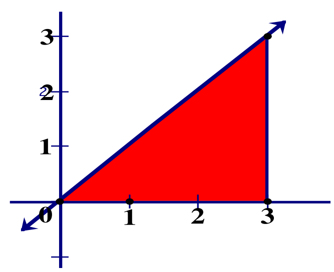
******* Area** $= \displaystyle\int_a^b (\textit{top equation} - \textit{bottom equation})\,dx$

EX #3: Find Area of each shaded region

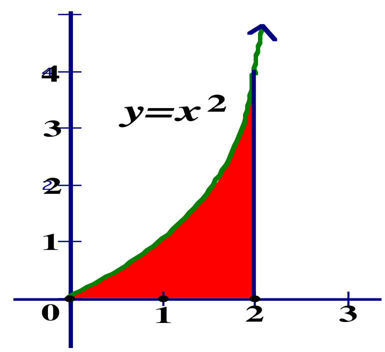
$a)$



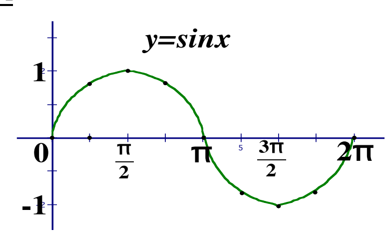
$b)$



$c)$



EX #4: Find Area of the region between $y = \sin x$ and the x-axis from $[0, 2\pi]$



4-4 The Fundamental Theorem of Calculus (Cont.)

******* Average Value** (use this when you are asked to find the average of anything)

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\textbf{Average value} = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx$$

EX#1: Find average value of $f(x)$ on the closed interval.

$a)\quad f(x)=x^2 \quad [0,3]$

$b)\quad f(x)=\sqrt{x} \quad [1,9]$

EX#2: Find the average velocity from $[0,2]$

$v(t)=3t^2+2t+5$

*****2nd Fundamental Theorem of Calculus** (When taking the derivative of an integral)

Plug in the variable on top times its derivative minus plug in the variable on bottom times its derivative.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Verification: $\dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt = f(x)\cdot 1 - f(0)\cdot 0 = f(x)$

EX: $\dfrac{d}{dx}\displaystyle\int_x^0 f(t)\,dt = f(0)\cdot 0 - f(x)\cdot 1 = -f(x)$

EX: $\dfrac{d}{dx}\displaystyle\int_x^{x^2} f(t)\,dt = f\left(x^2\right)\cdot 2x - f(x)\cdot 1$

EX #3: Evaluate each.

$a)\quad \dfrac{d}{dx}\displaystyle\int_2^x t^5\,dt =$

$b)\quad \dfrac{d}{dx}\displaystyle\int_x^2 t^5\,dt =$

$c)\quad F(x)=\displaystyle\int_1^{x^2}\left(t^4+7\right)dt =$

$F'(x)=$

$d)\quad F(x)=\displaystyle\int_x^{\sin x}\sqrt{1+t^2}\,dt =$

$F'(x)=$

4-5 Integration by Substitution

***Substitution** When integrating we usually let u = the part in the parenthesis, the part under the radical, the denominator, the exponent, or the angle of the trig. function.

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u) \cdot du = F(u) + C = F(g(x)) + C$$

Let $u = g(x)$

$du = g'(x)\,dx$

EX#1: $\int 3x^2 (x^3 + 7)^9 \, dx =$

EX#2: $\int (\sin x)^4 \cos x \, dx =$

EX#3: $\int x^3 (x^4 + 10)^7 \, dx =$

EX#4: $\int x \cos x^2 \, dx =$

EX#5: $\int_0^2 x^2 (x^3 - 1)^3 \, dx =$

EX#6: $\int \frac{5x^5}{(x^6 + 2)^5} \, dx =$

EX#7: $\int x\sqrt{x+3} \, dx =$

EX#8: $\int x^3 \sqrt{x^2 - 4} \, dx =$

4-6 Numerical Integration (Approximating Area)

We approximate Area using rectangles (left, right, and midpoint) and trapezoids.

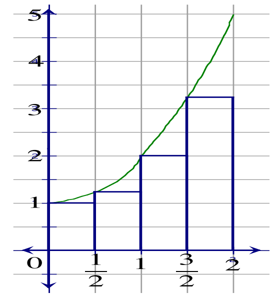
*Riemann Sums

$a)$ **Left edge Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)

$$A = \left(\frac{b-a}{n}\right)(\text{ left height of each rectangle })\qquad A = \left(\frac{2-0}{4}\right)\left(1+\frac{5}{4}+2+\frac{13}{4}\right)$$

Total Area= $\dfrac{30}{8} \doteq 3.750$

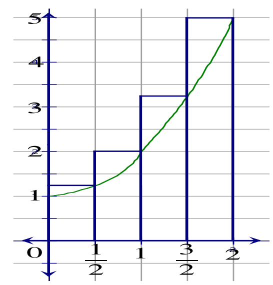


$b)$ **Right edge Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)

$$A = \left(\frac{b-a}{n}\right)(\text{ right height of each rectangle })\qquad A = \left(\frac{2-0}{4}\right)\left(\frac{5}{4}+2+\frac{13}{4}+5\right)$$

Total Area = $\dfrac{46}{8} \doteq 5.750$

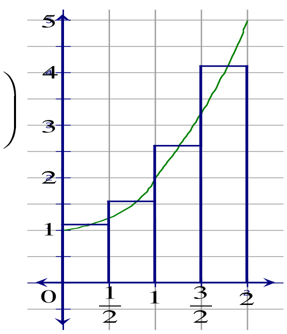


$c)$ **Midpoint Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)

$$A = \left(\frac{b-a}{n}\right)(\text{ midpt. height of each rectangle })\qquad A = \left(\frac{2-0}{4}\right)\left(\frac{17}{16}+\frac{25}{16}+\frac{41}{16}+\frac{65}{16}\right)$$

Total Area = $\dfrac{148}{32} \doteq 4.625$



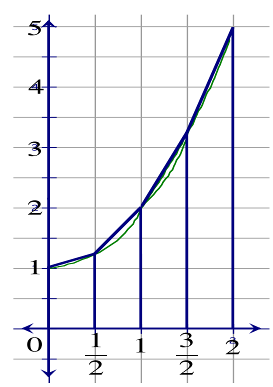
*Trapezoidal Rule

(used to approximate area under a curve, using trapezoids).

$$Area \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3).....2f(x_{n-1})+f(x_n)\right]$$

where n is the number of subdivisions.

$d)$ $f(x)=x^2+1$ Approximate the area under the curve from $[\,0,2\,]$ using the trapezoidal rule with 4 subdivisions.



$$A = \frac{2-0}{2(4)}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+f(2)\right]$$

$$= \frac{1}{4}\left[1+2\left(\frac{5}{4}\right)+2(2)+2\left(\frac{13}{4}\right)+5\right]$$

$$= \frac{1}{4}\left[\frac{76}{4}\right] = \frac{76}{16} = 4\frac{3}{4} = 4.750$$

All you are doing is finding the area of the 4 trapezoids and adding them together!

$e)$ **Actual Area** $= \displaystyle\int_0^2 \left(x^2+1\right)dx = \frac{x^3}{3}+x\,\Big|_0^2 = \left(\frac{8}{3}+2\right)-(0+0) = \frac{14}{3} = 4.667$

*Approximating Area when given data only (no equation given)

To estimate the area of a plot of land, a surveyor takes several measurements. The measurements are taken every 15 feet for the 120 ft. long plot of land, where y represents the distance across the land at each 15 ft. increment.

x	0	15	30	45	60	75	90	105	120
y	58	63	72	60	62	69	61	74	67

$a)$ **Estimate using Trapezoidal Rule**

$$A \doteq \frac{120-0}{2(8)}\left[f(0)+2f(15)+.....+2f(105)+f(120)\right]$$

$$A \doteq \frac{15}{2}\left[58+126+144+120+124+138+122+148+67\right]$$

$$A \doteq 7852.5$$

$b)$ **Estimate using 4 Midpoint subdivisions**

$$A \doteq \frac{120-0}{4}\left[f(15)+f(45)+f(75)+f(105)\right]$$

$$A \doteq 30\left[63+60+69+74\right]$$

$$A \doteq 7980$$

$c)$ **Estimate Avg. value using Trapezoidal Rule**

$$\text{Avg.Value} \doteq \frac{1}{120}(7852.5) \doteq 65.4375$$

$d)$ **What are you finding in part c?**

The average distance across the land.

$e)$ **Estimate using Left Endpoint**

$$A \doteq \frac{120-0}{8}\left[f(0)+f(15)+f(30)+.....+f(105)\right]$$

$$A \doteq 15\left[58+63+72+60+62+69+61+74\right]$$

$$A \doteq 7785$$

$f)$ **Estimate using Right Endpoint**

$$A \doteq \frac{120-0}{8}\left[f(15)+f(30)+f(45).....+f(120)\right]$$

$$A \doteq 15\left[63+72+60+62+69+61+74+67\right]$$

$$A \doteq 7920$$

*Approximating Area when given data only (no equation given)

Unequal subdivisions: You must find each Area separately.

x	0	2	5	10
y	10	13	11	15

***a*) Estimate using Trapezoids** $\left(A=\frac{1}{2}(b_1+b_2)h\right)$

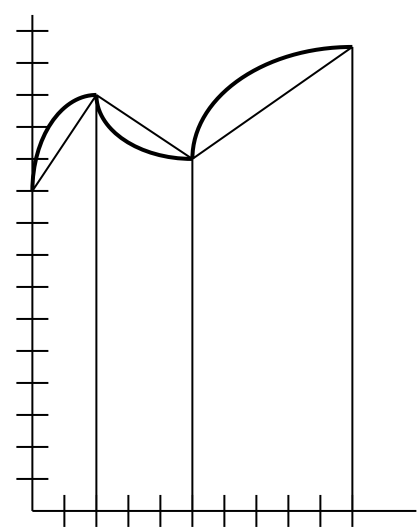
$$A \doteq \frac{1}{2}\cdot(10+13)\cdot 2 \;+\; \frac{1}{2}\cdot(13+11)\cdot 3 \;+\; \frac{1}{2}\cdot(11+15)\cdot 5 \quad \doteq 124$$

***b*) Estimate using Left Endpoint** $(A = width \cdot left\ height)$

$$A \doteq 2(10)+3(13)+5(11) \qquad \doteq 114$$

***c*) Estimate using Right Endpoint** $(A = width \cdot right\ height)$

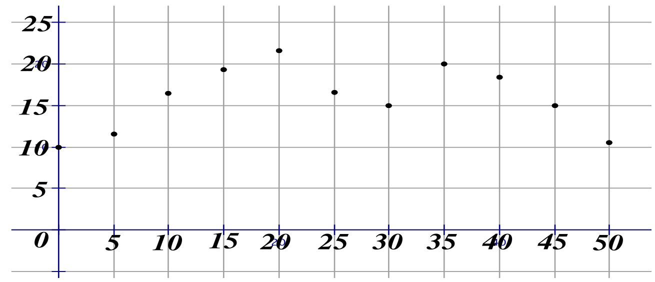
$$A \doteq 2(13)+3(11)+5(15) \qquad \doteq 134$$



Trapezoids shown

EX #1: To estimate the surface area of his pool, a man takes several measurements. The measurements are taken every 5 feet for the 50 ft. long pool, where y represents the distance across the pool at each 5 ft. increment.

x	0	5	10	15	20	25	30	35	40	45	50
y	10	12	17	19	22	17	15	20	18	15	11



Estimate each Area using 10 subdivisions

a) Use Trapezoidal Rule

b) Estimate Avg. value using Trapezoidal Rule

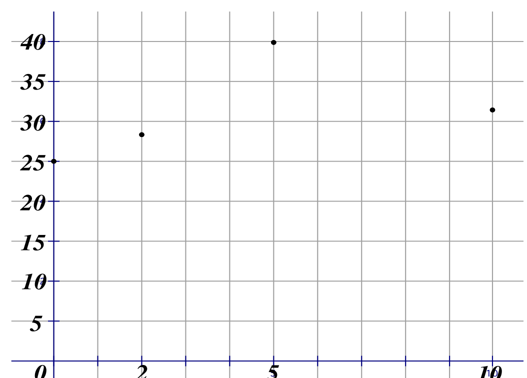
c) Use Right Endpoint

d) Use Left Endpoint

e) Use Midpoint with 5 subdivisions

EX #2:

x	0	2	5	10
y	25	28	40	32



Estimate each Area using 3 unequal subdivisions

a) Find Trapezoidal sum

b) Estimate Avg. value using Trapezoids

c) Find Right Endpoint sum

d) Find Left Endpoint sum

CH.5 Logarithmic, Exponential, and other Transcendental Functions

5-1 The Natural Logarithm Function and Differentiation

Properties of Logarithms

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \qquad e =$ natural number

logarithmic form $\Leftrightarrow$ **exponential form**

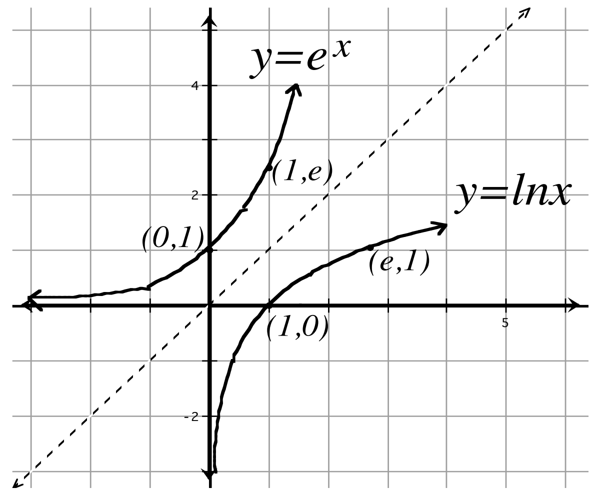
$$y=\ln x \quad \Leftrightarrow \quad e^y = x$$

Log Laws:

$y=\ln x^a \Leftrightarrow y = a\cdot \ln x \qquad \ln 8 = \ln 2^3 = 3\ln 2$

$\ln x + \ln y = \ln(xy) \qquad \ln 2 + \ln 5 = \ln 10$

$\ln x - \ln y = \ln\left(\frac{x}{y}\right) \qquad \ln 7 - \ln 2 = \ln\left(\frac{7}{2}\right)$



Memorize these graphs.
They are inverses of each other so they are symmetric about the line y=x.

Memorize: $\ln e = 1 \quad \ln 1 = 0$

Change of Base Law:

$y=\log_a x \quad \Rightarrow \quad y = \frac{\ln x}{\ln a}$ or $\frac{\log x}{\log a}$

Fact: You can't take a ln/log of a negative # or zero.

We use logarithms to solve any problem that has a variable in the exponent.

EX: $e^{5x} = 24 \quad \Rightarrow$ Take ln of both sides $\Rightarrow \quad \ln e^{5x} = \ln 24 \quad \Rightarrow \quad 5x\ln e = \ln 24 \quad \Rightarrow \quad x = \frac{\ln 24}{5}$

EX: $\ln x = 3 \quad \Rightarrow$ Take e to the power of both sides $\Rightarrow \quad e^{\ln x} = e^3 \quad \Rightarrow \quad x = e^3$

*Derivative of Natural Log

(2 STEPS: **1 divided by function • Derivative of function**)

$y = \ln(f(x)) \qquad y' = \frac{1}{f(x)}\cdot f'(x)$

Find derivative of each

EX#1: $\frac{d}{dx}\ln(x) =$

EX#2: $\frac{d}{dx}\ln(x^2) =$

EX#3: $\frac{d}{dx}\ln(x^3-1) =$

EX#4: $\frac{d}{dx}\log x =$

EX#5: $f(x) = \ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}} =$

5-2 The Natural Logarithmic Function: Integration

*** $\int \frac{f'(x)}{f(x)}\,dx = ln\left|f(x)\right|+C \quad \Rightarrow$ If the top is the derivative of the bottom, the answer is: $\ln|\text{bottom}|+\text{C}$.

*** $\int \frac{1}{x}\,dx \quad = ln|x|+C$

Integrals of other four trig. functions (All four of these are natural logarithm integrals.)

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx \boxed{= -\ln|\cos x|+C} \quad \text{or} \quad \boxed{= \ln|\sec x|+C}$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx \boxed{= \ln|\sin x|+C}$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx \boxed{= \ln|\sec x+\tan x|+C}$$

$$\int \csc x\,dx = \int \frac{\csc x(\csc x+\cot x)}{\csc x+\cot x}\,dx = \int \frac{\csc^2 x+\csc x\cot x}{\csc x+\cot x}\,dx$$

$$= -\int \frac{-(\csc^2 x+\csc x\cot x)}{\csc x+\cot x}\,dx \boxed{= -\ln|\csc x+\cot x|+C}$$

EX#1: $\int \frac{3x^2}{x^3-5}\,dx =$

EX#2: $\int \frac{x^3}{x^4+1}\,dx =$

EX#3: $\int \frac{1}{1-6x}\,dx =$

EX#4: $\int \frac{\sec^2 x}{\tan x}\,dx =$

EX#5: $\int_{-6}^{-2} \frac{1}{x}\,dx =$

EX#6: $\int \frac{2x}{\left(x^2+1\right)^7}\,dx =$

EX#7: $\int \frac{x^2+x+1}{x^2+1}\,dx =$

EX: $\int \tan 3x\,dx = -\frac{1}{3}\ln|\cos 3x|+C$

EX: $\int \sec 7x\,dx = \frac{1}{7}\ln|\sec 7x+\tan 7x|+C$

EX: $\int \csc 4x\,dx = -\frac{1}{4}\ln|\csc 4x+\cot 4x|+C$

EX: $\int \cot 9x\,dx = \frac{1}{9}\ln|\sin 9x|+C$

5-3 Inverse Functions

INVERSES: To find an inverse, $f^{-1}(x)$, you switch the $x's$ and $y's$ and solve for y.

EX: $f(x)=2x+3$ Find $f^{-1}(x)$.

Inverse: $x=2y+3$

$\dfrac{x-3}{2}=y$ so $f^{-1}(x)=\dfrac{x-3}{2}$

EX: $f(x)=e^{x}$ Find $f^{-1}(x)$.

Inverse: $x=e^{y} \Rightarrow \ln x=\ln e^{y}$

$\ln x=y$ so $f^{-1}(x)=\ln x$

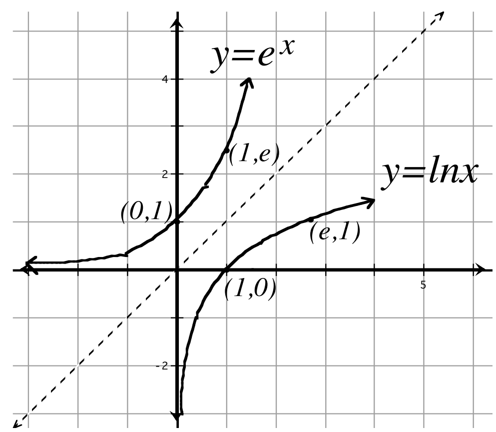
Facts about inverses

1) When you plug one inverse into the other you always get the answer x and vice versa.

$(f\circ f^{-1})(x)=f\left(f^{-1}(x)\right)=x$ and $(f^{-1}\circ f)(x)=f^{-1}\left(f(x)\right)=x$

EX: $f(x)=e^{x}$ $f^{-1}(x)=\ln x$

$(f\circ f^{-1})(x)=e^{\ln x}=x$ $(f^{-1}\circ f)(x)=\ln e^{x}=x$



2) Graphs that are inverses are symmetric about the line $y=x$.

*** $\dfrac{dy}{dx}$ finds the slope of the tangent line.

$$\frac{dx}{dy}=\frac{1}{dy/dx}$$

*** $\dfrac{dx}{dy}$ finds the slope of the tangent line of the inverse.

Formula for finding derivative of the inverse at a point: $g'(y)=\dfrac{1}{f'(x)}$ (where g is the inverse of f)

EX #1: Let $y=x^{3}+x$. If h is the inverse function of f, then $h'(2)=$

$A)\ \dfrac{1}{13}$ $B)\ \dfrac{1}{4}$ $C)\ 1$ $D)\ 4$ $E)\ 13$

EX #2: Let f be a differentiable function such that $f(4)=20$, $f(8)=4$, $f'(4)=-7$, and $f'(8)=-5$. The function g is differentiable and $g(x)=f^{-1}(x)$ for all x. What is the value of $g'(4)$?

$A)\ \dfrac{-1}{5}$ $B)\ \dfrac{-1}{7}$ $C)\ \dfrac{1}{8}$ $D)\ \dfrac{1}{4}$

$E)$ The value of $g'(4)$ cannot be determined from the information given.

5-4 Exponential Functions: Differentiation and Integration

Trig. Review

$e^{\ln x} =$ $\qquad$ $e^{\ln 3} =$ $\qquad$ $e^{2\ln 5} =$

Exponential Differentiation $\left(\underline{\text{Constant}^{\text{Variable}}}\right)$

$\dfrac{d}{dx}e^x = e^x \cdot 1 \cdot \ln e = e^x$ $\qquad$ **(3 steps: itself, multiplied by deriv. of exponent, multiplied by ln of base)**

$\dfrac{d}{dx}e^{2x} =$

$\dfrac{d}{dx}e^{5x^3} =$

Exponential Integration $\left(\underline{\text{Constant}^{\text{Variable}}}\right)$

$\int e^x\, dx = \dfrac{e^x}{1 \cdot \ln e} + C = e^x + C$ $\qquad$ **(3 steps: itself, divided by deriv. of exponent, divided by ln of base)**

$\int e^{2x}\, dx =$

$\int e^{5x^3}\, dx =$

EX#1: Find derivative of each

$a)\quad y = e^{7x^2}$ $\qquad$ $b)\quad y = e^x \sin x$

EX#2: Evaluate each integral

$a)\quad \int \dfrac{e^x}{e^x + 1}\, dx =$ $\qquad$ $b)\quad \int_0^{\ln 2} e^{3x}\, dx =$

EX#3: Evaluate each integral

$a)\quad \int x^2 \cdot e^{5x^3}\, dx =$ $\qquad$ $b)\quad \int x \cdot e^{3x^2}\, dx =$

5-5 Bases other than e and Applications

***Derivative of Constant$^{\text{Variable}}$** $\quad y = a^{f(x)} \qquad y' = a^{f(x)} \cdot f'(x) \cdot \ln a$

(3 steps: itself, multiplied by derivative of exponent, multiplied by ln of base)

EX#1: $\dfrac{d}{dx} 7^x =$

EX#2: $\dfrac{d}{dx} 3^{x^2} =$

EX#3: $f(t) = t^2 \cdot 5^t$

$f'(t) =$

***Integral of Constant$^{\text{Variable}}$** $\quad \int a^x \, dx = \dfrac{a^x}{1 \cdot \ln a} + C$

(3 steps: itself, divided by deriv. of exponent, divided by ln of base)

EX#4: $\int 9^x \, dx =$

EX#5: $\int 5^{4x} \, dx =$

EX#6: $\int 8^{x^2} \, dx =$

EX#7: $\int x \cdot 8^{x^2} \, dx =$

***Derivative of Variable$^{\text{Variable}}$** $\quad y = f(x)^{g(x)}$

(take **ln** of both sides then take derivative of both sides)

$\ln y = g(x) \ln f(x) \qquad \dfrac{1}{y}\dfrac{dy}{dx} = g'(x) \ln f(x) + \dfrac{f'(x)}{f(x)} g(x) \qquad \dfrac{dy}{dx} = f(x)^{g(x)} \left(g'(x) \ln f(x) + \dfrac{f'(x)}{f(x)} g(x) \right)$

EX#8: $y = x^{\sin x}$

EX#9: $y = (2x)^{5x}$

***Derivative of Variable$^{\text{Variable}}$** $\quad y = f(x)^{g(x)}$ **(alternate way)**

Need to change $y = f(x)^{g(x)}$ to $y = e^{\ln f(x)^{g(x)}} \Rightarrow y = e^{g(x) \ln f(x)}$ then take derivative.

EX: $\quad y = x^{\sin x} \Rightarrow y = e^{\sin x \ln x}$

$$y' = e^{\sin x \ln x} \cdot \left[\cos x \cdot \ln x + \frac{1}{x} \sin x \right] = x^{\sin x} \left[\cos x \cdot \ln x + \frac{\sin x}{x} \right]$$

***Derivative of Variable$^{\text{Variable}}$** (Shortcut)

(2 steps: itself, multiplied by product rule of exponent and ln of base)

5-6 Inverse Trigonometric Functions: Differentiation

$\sin\theta = \frac{1}{2}$ means give me all the answers. $\left\{ \begin{aligned} \theta &= \frac{\pi}{6} + 2n\pi \\ &= \frac{5\pi}{6} + 2n\pi \; ; \; n \in Z \end{aligned} \right\}$

$\sin\theta = \frac{1}{2}$; $[0, 2\pi]$ means give me only the answers from $[0, 2\pi]$. $\left\{ \theta = \frac{\pi}{6}, \frac{5\pi}{6} \right\}$

$\arcsin\frac{1}{2}$ means give me the principal (first) answer only between $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$. $\left\{ \theta = \frac{\pi}{6} \right\}$

EX #1: Trig. Review

$a)$ $\arcsin\frac{1}{2} =$ $b)$ $\arcsin\left(-\frac{1}{2}\right) =$ $c)$ $\arcsin\left(\frac{1}{\sqrt{2}}\right) =$ $d)$ $\arcsin\left(-\frac{1}{\sqrt{2}}\right) =$

$e)$ $\arctan 1 =$ $f)$ $\arctan(-1) =$ $g)$ $\arcsin 1 =$ $h)$ $\arcsin(-1) =$

$i)$ $\arctan 0 =$ $j)$ $\arcsin 0 =$ $k)$ $\arctan\sqrt{3} =$ $l)$ $\arcsin\left(-\frac{\sqrt{3}}{2}\right) =$

*Inverse Trig. Functions

$y = \arcsin f(x)$ $\quad y' = \frac{1}{\sqrt{1-(f(x))^2}} \cdot f'(x)$ $\quad y = \arccos f(x)$ $\quad y' = \frac{-1}{\sqrt{1-(f(x))^2}} \cdot f'(x)$

$y = \arctan f(x)$ $\quad y' = \frac{1}{1+(f(x))^2} \cdot f'(x)$ $\quad y = \operatorname{arccot} f(x)$ $\quad y' = \frac{-1}{1+(f(x))^2} \cdot f'(x)$

$y = \operatorname{arcsec} f(x)$ $\quad y' = \frac{1}{|f(x)|\sqrt{(f(x))^2-1}} \cdot f'(x)$ $\quad y = \operatorname{arccsc} f(x)$ $\quad y' = \frac{-1}{|f(x)|\sqrt{(f(x))^2-1}} \cdot f'(x)$

Take derivative of each

EX#1: $y = \arcsin 3x^4$ **EX#2:** $y = \arctan 3x^4$ **EX#3:** $y = \operatorname{arcsec} 6x$

EX#4: $y = \arcsin e^x$ **EX#5:** $y = \arctan(\ln x)$ **EX#6:** $y = \operatorname{arcsec}(\sin x)$

5-7 Inverse Trigonometric Functions: Integration

*Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+C \quad \text{OR} \quad = -\arccos\frac{x}{a}+C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+C \quad \text{OR} \quad = -\frac{1}{a}\operatorname{arccot}\frac{x}{a}+C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\frac{|x|}{a}+C \quad \text{OR} \quad = -\frac{1}{a}\operatorname{arccsc}\frac{|x|}{a}+C$$

Find variable v and constant a. The top MUST be the derivative of the variable v.

EX#1: $\int \frac{1}{\sqrt{9-x^2}}\,dx =$

EX#2: $\int \frac{1}{x^2+16}\,dx =$

EX#3: $\int \frac{1}{x\sqrt{x^2-25}}\,dx =$

EX#4: $\int \frac{12}{\sqrt{49-4x^2}}\,dx =$

EX#5: $\int \frac{5}{9x^2+36}\,dx =$

EX#6: $\int \frac{10}{x\sqrt{16x^2-25}}\,dx =$

EX#7: $\int \frac{7}{x\sqrt{9x^4-4}}\,dx =$

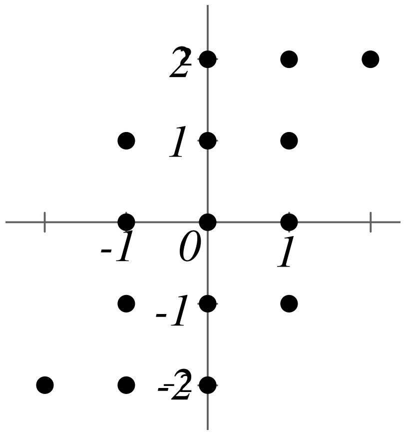
EX#8: $\int \frac{x}{x^2+81}\,dx =$

CH. 6 Differential Equations

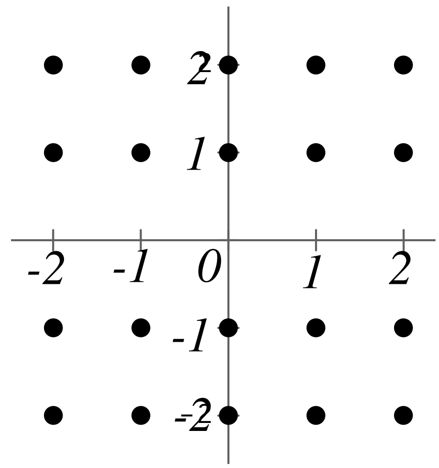
6-1 Slope Fields and Euler's Method

Draw the slope field for each

EX #1: $\frac{dy}{dx} = x - y$

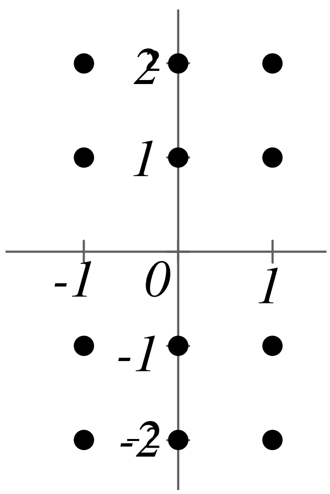


EX #2: $a)$ Draw slope field $\frac{dy}{dx} = \frac{-x}{y}$



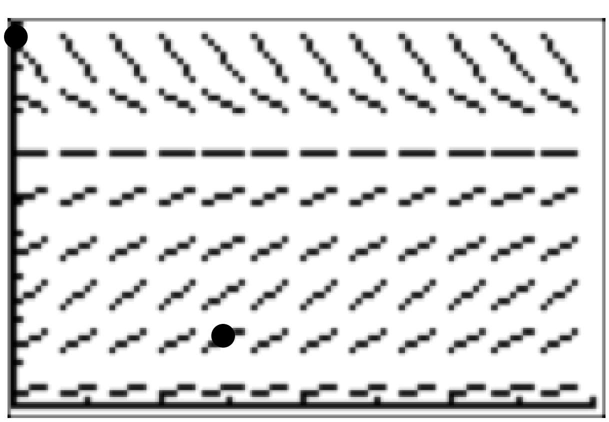
$b)$ Find the particular solution $y = f(x)$ for the given differential equation at the point $(0,-1)$.

EX #3: Draw slope field for $\frac{dy}{dx} = \frac{-2x}{y}$

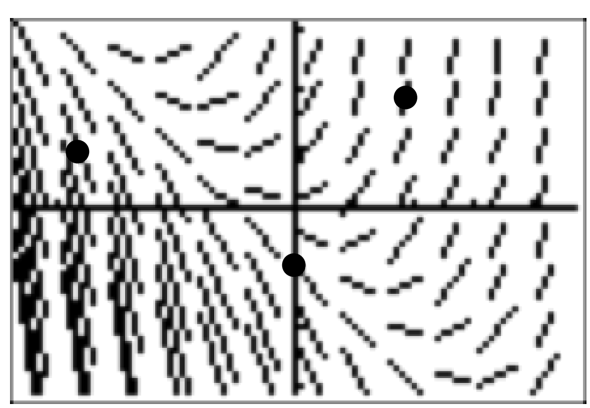


Here are the slope fields for the given differential equations. Sketch the solution for each given point.

EX #4: $\frac{dy}{dx} = \frac{y}{8}(6-y)$; $(0,9)$ $(3,2)$



EX #5: $\frac{dy}{dx} = x + y$; $(-4,2)$ $(0,-2)$ $(2,3)$



6-2 Differential Equations: Growth and Decay

***Growth Formula (Can be used at any time)**

$$y = C \cdot e^{kt} \qquad (\text{Comes from } y' = ky)$$

***Compound Continuous Formula (Money equation)**

$$A = P \cdot e^{rt}$$

***Double - Life Formula (Use only when doubling is mentioned)**

$$y = C \cdot (2)^{t/d}$$

***Half - Life Formula (Use only when half - life is mentioned)**

$$y = C \cdot \left(\frac{1}{2}\right)^{t/h}$$

A, y	=	ending amount
C, P	=	initial amount
t	=	time
d	=	double-life time
h	=	half-life time
k	=	growth constant

EX #1: A certain kind of algae doubles every 3 days. If the beginning population of the algae is 800, what will the population be after 1 week?

EX #2: If I invest \$10,000 compounded continuously for 20 years and it grows to \$100,000. At what rate was the money invested?

EX #3: SHHS population in 1980 was 1200 people. SHHS population in 2000 was 1900 people. What will the population be in 2015 at the same growth rate? (Round answer to nearest whole #)

EX #4: Carbon has a half-life of 5730 years. We measure the amount of carbon in a tree and it has 30% less carbon than when it was planted. How old is the tree? (Round answer to nearest whole #)

6-3 Separation of Variables and the Logistic Equation

DIFFERENTIAL EQUATIONS (**Separating Variables**) (used when you are given the derivative and you need to find the original equation. We separate the x's and y's and take the integral).

EX #1: Find the general solution given

$a)\ \dfrac{dy}{dx}=\dfrac{x^2+2}{3y^2}$

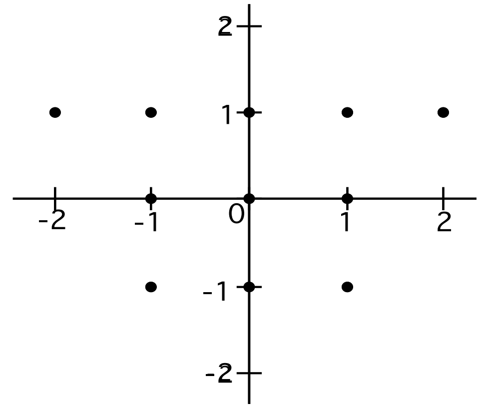
$b)\ yy'-2e^x=0$

EX #2: Find the particular solution $y=f(x)$ of $\dfrac{dy}{dx}-2xy=0 \qquad y(0)=10$

EX #3: AP Test 2000 BC#6 $\dfrac{dy}{dx}=x(y-1)^2$

$a)$ Find the particular solution $y=f(x)$ given $f(0)=-1$

$b)$ Draw slope field at 11 points indicated.



If the rate of growth of something is proportional to itself $(y'=ky)$, then it is the growth formula $\left(y=C_1e^{kt}\right)$.

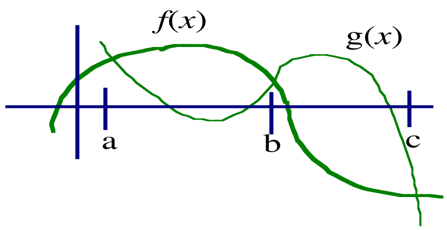
Proof: $y'=ky \Rightarrow \dfrac{dy}{dt}=ky \Rightarrow \dfrac{dy}{y}=k\,dt \Rightarrow \displaystyle\int\frac{dy}{y}=\int k\,dt \Rightarrow \ln y=kt+C \Rightarrow$

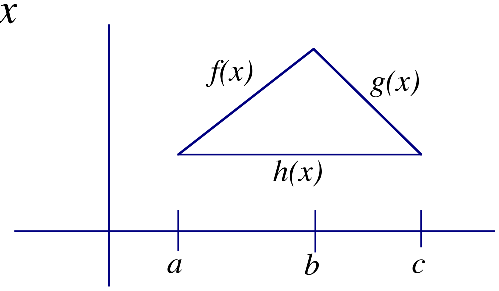
$e^{\ln y}=e^{kt+C} \Rightarrow y=e^{kt}\cdot e^C \Rightarrow y=C_1e^{kt}$

CH.7 Application of Integration

7-1 Area of a Region Between Two Curves

***Area** $A = \int_a^b [\text{top equation-bottom equation}]\, dx$





Area from *a* to *c*

$$A = \int_a^b [f(x) - g(x)]\, dx + \int_b^c [g(x) - f(x)]\, dx$$

Area from *a* to *c*

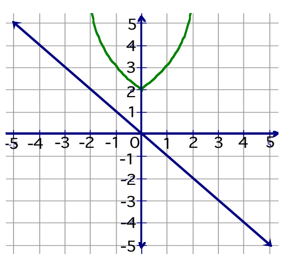
$$A = \int_a^b [f(x) - h(x)]\, dx + \int_b^c [g(x) - h(x)]\, dx$$

Steps to find Area :

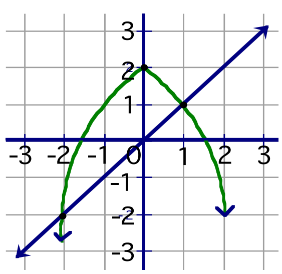
1) Find out where the equations meet.
2) Find out which equation is on top.

Find the Area of each enclosed region.

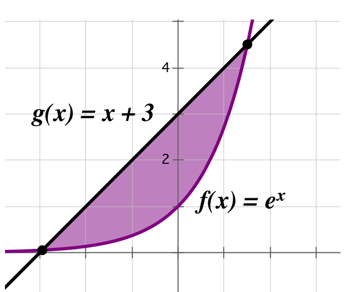
EX #1 : $f(x) = x^2 + 2 \quad g(x) = -x \quad [-1, 2]$.



EX #2 : $f(x) = x \quad g(x) = 2 - x^2$

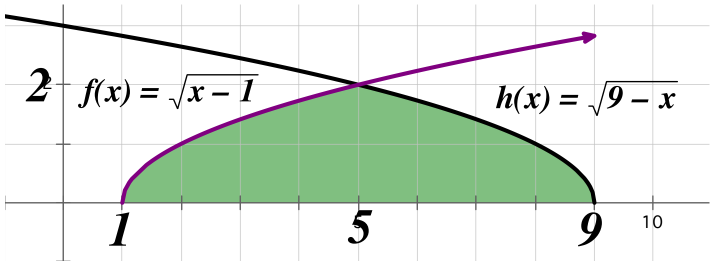


EX#3 : $f(x) = e^x \quad g(x) = x + 3$

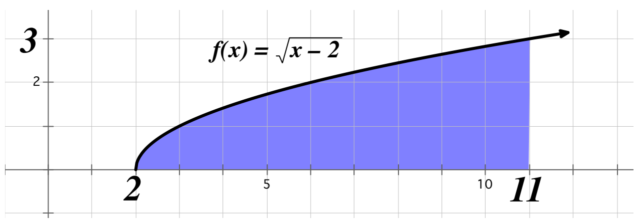


We can find Area using vertical (top to bottom) or horizontal (right to left) cross sections.

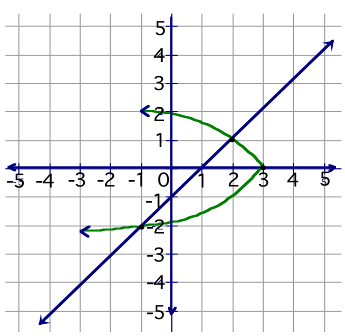
EX#4 : Find the Area of the shaded region bounded by the x-axis and the graphs of $f(x)$ and $g(x)$.



EX #5 : $.f(x)=\sqrt{x-2}$; $[2\ ,\ 11]$ Find the area of the shaded region.



EX #6 : Find the Area of the region bounded by the graphs of $x=3-y^2$ and $x=y+1$.



7-2 Volume: The Disc method

******Volume** $\quad V = \int A(x)\, dx$ **where $A(x)$ is the Area of the cross section.**

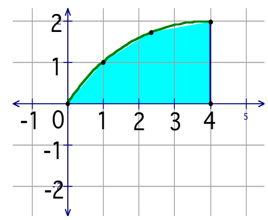
Volume if cross section rotated is a circle $\left(A = \pi r^2\right)$

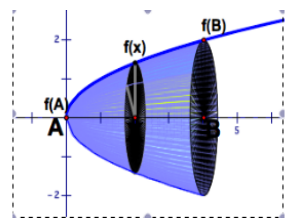
$$V = \pi \int_a^b \left[(\text{top function})^2 - (\text{bottom function})^2 \right] dx$$

radius

EX #1: $f(x) = \sqrt{x} \quad [0, 4]$

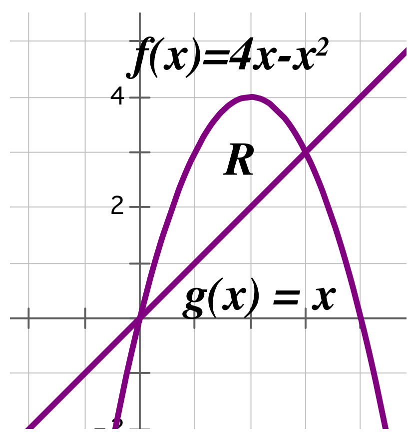
Find the volume of the enclosed region R between $f(x)$ and the x-axis rotated about the x-axis.





EX #2: $f(x) = 4x - x^2 \qquad g(x) = x$

Find the volume of the enclosed region rotated about the x-axis



7-3 Volume: The Shell method

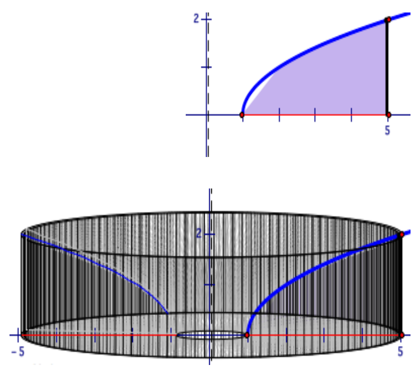
Volume if cross section rotated is a cylinder $(A = 2\pi rh)$

$$V = 2\pi \int_a^b x\left[\text{top function} - \text{bottom function}\right] dx$$

radius **height**

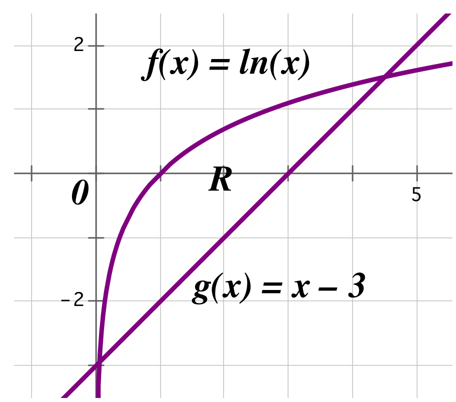
EX #1: $f(x) = \sqrt{x-1}$ $[1,5]$

Find the volume of the enclosed region between $f(x)$ and the x-axis rotated about the y-axis



EX #2: $f(x) = \ln x$ $g(x) = x - 3$

Find the volume of the enclosed region R revolved about the y-axis



Volumes: Rotations about other lines

***Volume rotated about:** (**Vertical cross section**)

the x - axis

$$\pi\int_a^b\left[\left(f(x)\right)^2-\left(g(x)\right)^2\right]dx$$

the y - axis

$$2\pi\int_a^b x\left[f(x)-g(x)\right]dx$$

the line $y=-k$

$$\pi\int_a^b\left[\left(f(x)+k\right)^2-\left(g(x)+k\right)^2\right]dx$$

the line $x=-c$

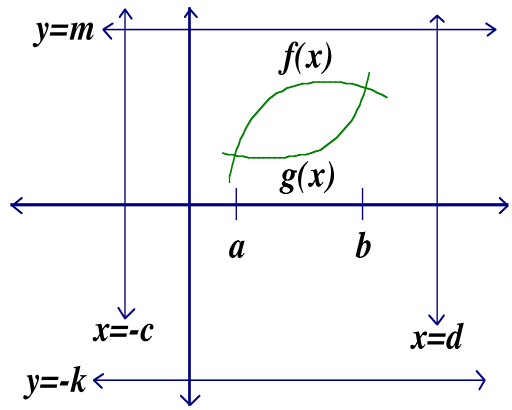
$$2\pi\int_a^b (x+c)\left[f(x)-g(x)\right]dx$$

the line $y=m$

$$\pi\int_a^b\left[\left(m-g(x)\right)^2-\left(m-f(x)\right)^2\right]dx$$

the line $x=d$

$$2\pi\int_a^b (d-x)\left[f(x)-g(x)\right]dx$$



EX #1: $f(x)=4x-x^2$ $\quad g(x)=x^2$ $\quad [0,2]$

Find the volume of the solid formed when rotating the enclosed region about the given lines.

the x - axis | **the line $y=-2$** | **the line $y=5$**

the y - axis | **the line $x=-3$** | **the line $x=6$**

EX #2: $f(x)=\sqrt{x}$ $\quad [0,4]$

Find the volume of the solid formed when rotating the enclosed region about the given lines.

the x - axis | **the line $y=-5$** | **the line $y=6$**

the y - axis | **the line $x=-3$** | **the line $x=4$**

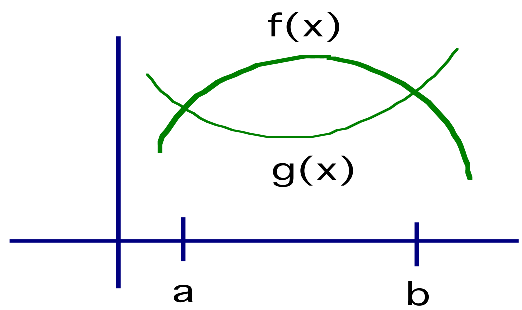
Volumes of known cross sections

***Volume (Region is not rotated)** $V = \int A(x)\,dx$ **where $A(x)$ is the Area of the cross section.**

- Sometimes we will find the volume of regions that have different cross sections (not a circle or a cylinder).
- These regions are not rotated but come out at us.
- We must first find the Area of the cross section, then take it's integral.

***** $V = \text{multiplier} \cdot \int_a^b (\textit{top equation - bottom equation})^2\,dx$ **for all shapes except possibly rectangles.**

EX: Let R be the region in the first quadrant below $f(x)$ and above $g(x)$ from $x = a$ to $x = b$. Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are :



Squares $\left(A = s^2\right)$ $V = \int_a^b (f(x) - g(x))^2\,dx$

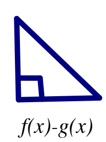
Equilateral Δ's $\left(A = \frac{s^2\sqrt{3}}{4}\right)$ $V = \frac{\sqrt{3}}{4}\int_a^b (f(x) - g(x))^2\,dx$

Semicircle $\left(A = \frac{\pi r^2}{2}\right)$ $V = \frac{\pi}{8}\int_a^b (f(x) - g(x))^2\,dx$

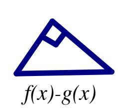
Rectangle with $h = 7 \cdot b$ $(A = bh = b \cdot 7b)$ $V = 7\int_a^b (f(x) - g(x))^2\,dx$

Rectangle with $h = 12 - x$ $(A = bh = b \cdot (12 - x))$ $V = \int_a^b (f(x) - g(x))(12 - x)\,dx$

Isosceles Right Triangle $\left(A = \frac{1}{2} b \cdot h\right)$ $V = \frac{1}{2}\int_a^b (f(x) - g(x))^2\,dx$ (Leg is base)



$V = \frac{1}{4}\int_a^b (f(x) - g(x))^2\,dx$ (Hyp. is base)



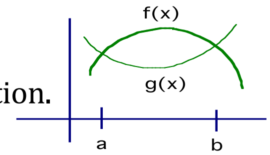
Multipliers for other figures:

30-60-90(SL): $\frac{\sqrt{3}}{2}$ 30-60-90(LL): $\frac{1}{2\sqrt{3}}$ 30-60-90(HYP): $\frac{\sqrt{3}}{8}$

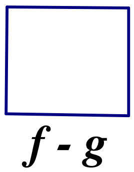
Regular Hexagon: $\frac{3\sqrt{3}}{2}$ Regular Polygon: $\frac{\text{\# of sides}}{4} \cdot \tan(\text{half of the interior angle})$

Volumes of known cross sections

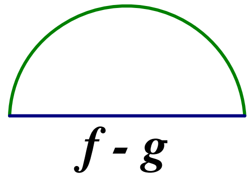
EX: Find the area of each figure in terms of f and g where f is the top equation and g is the bottom equation.



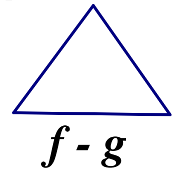
$a)$ square



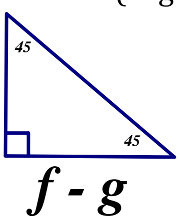
$b)$ semicircle



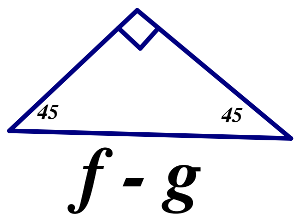
$c)$ equilateral △



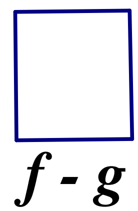
$d)$ isos. rt. △ (Leg is base)



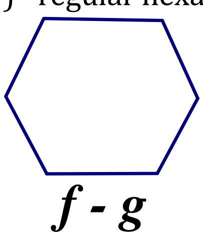
$e)$ isos. rt. △ (Hyp. is base)



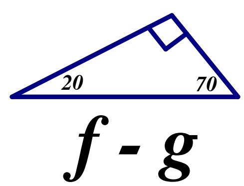
$f)$ rectangle with height $12-x$



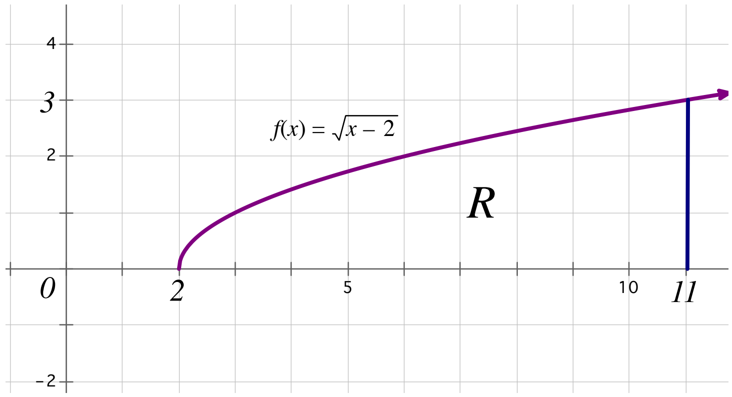
$g)$ regular hexagon



$h)$ $20°-70°-90°$ △ (Hyp. is base)



EX#1: Let R be the region in the first quadrant under the graph of $y=\sqrt{x-2}$ for $2\le x\le 11$. Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis (vertical cross sections) are :



$a)$ Squares

$b)$ Equilateral Δ's

$c)$ Semicircle

$d)$ Rectangle with $h=17\cdot b$

$e)$ Rectangle with $h=8-x$

$f)$ Isos.Rt.▵ (Leg is base)

Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the y-axis (horizontal cross sections) are :

$g)$ Squares

$h)$ Isos.Rt.▵ (Hypotenuse is base)